Math 6302 Homework Solutions

25) (i) Let $p(x) = \|x\|$. Then $p(x+y) = \|x+y\| \leq \|x\| + \|y\| = p(x) + p(y)$. Also, if $\alpha \geq 0$, then $p(\alpha x) = \|\alpha x\| = |\alpha|\|x\| = \alpha\|x\| = \alpha p(x)$. It follows that p is a sublinear functional.
(ii) X is not empty, $0 \in X$. Let $x \in X$, then$p(0) = p(0x) = 0p(x) = 0$. Also, $0 = p(0) = p(x-x) \leq p(x) + p(-x)$. It follows that $p(-x) \geq -p(x)$.

26) Let $Z = \{x \in X : x = \alpha x + 0, \alpha \in R\}$ and define the linear functional f on Z by $f(x) = \alpha p(x_0)$. Then $f(x) \leq p(x)$. By the Hahn-Banach Theorem there exists a $\bar{f}$ bounded linear functional defined on X such that $\bar{f}(x) \leq p(x)$. Clearly, $-\bar{f}(x) = \bar{f}(-x) \leq p(x)$. It follows that $-p(-x) \leq \bar{f}(x) \leq p(x)$.

27) Let $\bar{f}(x) = \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3$. Then $\|\bar{f}\|(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)^{\frac{1}{2}} \geq \|f\|$ and $\|\bar{f}\| = \|f\|$ iff $\alpha_3 = 0$.

28) let $\bar{f} = \langle x, \frac{x_0}{\|x_0\|}\rangle$. Then $\|\bar{f}\|\|\frac{x_0}{\|x_0\|}\| = 1$ and $\bar{f}(x_0) = \langle x_0, \frac{x_0}{\|x_0\|}\rangle = \|x_0\|$.

29) (i) $((\alpha T)^x g)(x) = g((\alpha T)x) = g(\alpha Tx) = \alpha g(Tx) = \alpha(t^x g)(x) = ((\alpha T^x)g)(x)$.
(ii) We have $(ST)^x = T^x S^x$ because $((ST)^x g)(x) = g((ST)(x)) = g(S(T(x))) = (s^x g)(Tx) = T^x((S^x g)(x)) = (T^x(S^x g))(x) = ((T^x S^x)g)(x)$.
It follows that $(T^n)^x = (T^{n-1})^x T^x = ... = (t^x)^n$..

30) (i) Let $g_x(f) = f(x)$, where $x \in X$ is fixed. Then $|g_x(f)| = |f(x)| \leq \|f\|\|x\|$ and therefore g is bounded and $\|g\| \leq \|x\|$. Let $\bar{f} \in X'$ such that $\|\bar{f}\| = 1$ and $\bar{f}(x) = \|x\|$. (We established the existence of such $\bar{f}$.) Then $\|g_x\| \geq \frac{|g_x(\bar{f})|}{\|\bar{f}\|} = \|x\|$. It follows that $\|g_x\| = \|x\|$.
(ii) Consider $C : x \to X''$, where $x \to g_x$. Then C is linear, because $\forall f \in X'$ we have

$$g_{\alpha x+\beta y}(f) = f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) = \alpha g_x(f) + \beta g_y(f).$$

Then $C(\alpha x+\beta y) = \alpha g_x+\beta g_y = \alpha C(x)+\beta C(y)$. Also, $\|C(x)\| = \|g_x\| = \|x\|$, and then $\|g_x - g_y\| = \|g_{x-y}\| = \|x - y\|$ and C is isometric. Moreover, if $x \neq y$, then $g_x \neq g_y$ and therefore C is injective. It follows that C is an isomorphism onto its range.
(iii) X'' is complete being the dual of X'. By assumption X is reflexive,

hence $R(C) = X''$. Part (ii) implies the completeness of X via isomorphism.

31) Let X be the normed space of all polynomials with norm $\|x\| = \max_i |\alpha_i|$ (the α_i's are the coefficients). Define the linear functional f_n by $f_n(x) = \alpha_0 + \alpha_1 + ... + \alpha_{n-1}$. Then $|f_n(x)| \leq n\|x\|$. Also for each fixed x we have $|f_n(x)| \leq c_x$. On the other hand, for $x(t) = 1 + t + t^2 + ... + t^n$, we have $\|x\| = 1$ and $f_n(x) = n\|x\|$. Hence $\|f_n\| \geq \frac{|f_n(x)|}{\|x\|} = n$. It follows that the sequence $(\|f_n\|)$ is unbounded. The Uniform Boundedness Theorem implies that X is not complete.

32) Recall that in a Banach space X if a sequence (x_n) is such that $(f(x_n))$ is bounded for all $f \in X'$, then $(\|x_n\|)$ is bounded and therefore (iii) implies (ii). Also, (ii) implies (i) by the Uniform Boundedness Theorem. Finally, (i) implies (iii) since $|g(T_n x)| \leq \|g\|\|T_n\|\|x\|$.

33) Let (x_n) be any weak Cauchy sequence in X. Then $(f(x_n))$ converges for every $f \in X'$. For $x_n \in X$ there is a $g_{x_n} \in X''$ such that $f(x_n) = g_{x_n}(f)$. Hence $(g_{x_n}(f))$ converges, say, $g_{x_n}(f) \to g(f)$. Weak Cauchyness of (x_n) implies the boundedness of (x_n) and then since $\|g_{x_n}\| = \|x_n\|$ we have that g is bounded. Also, g is linear and therefore $g \in X''$. Since X is reflexive, there is an x such that $g(f) = f(x)$. Hence $f(x_n) \to f(x)$. Since $f \in X'$ was arbitrary, this shows that (x_n) converges to x weakly. Since (x_n) was any weak Cauchy sequence, X is weakly complete.

34) (i) If $R(T)$ is closed in Y, it is complete, and boundedness follows from the Open Mapping Theorem.
(ii) Assume T^{-1} to be bounded, $y \in \overline{R(T)} \subset Y$, (y_n) in $R(T)$ such that $y_n \to y$, and $x_n = T^{-1}y_n$. Since T^{-1} is continuous and X is complete, (x_n) converges, say, $x_n \to x$. Since T is continuous, $y_n = Tx_n \to Tx$. Hence $y = Tx \in R(T)$, so that $R(T)$ is closed because $y \in \overline{R(T)}$ was arbitrary.

35) (i) Consider any $a \in \bar{A}$. Let $a_n \to a$, where $a_n \in A$. Let $c_n \in C$ be such that $a_n = Tc_n$. Since C is compact, (c_n) has a subsequence (c_{n_k}) which converges, say, $c_{n_k} \to c \in C$. Also $Tc_{n_k} \to a$, and $Tc = a \in A$ because T is closed by assumption.
(ii) Consider any $b \in \bar{B}$. Let $b_n \to b$, where $b_n \in B$. Let $k_n = Tb_n$. Since K is compact, (k_n) has a subsequence (k_{n_i}) which converges, say, $k_{n_i} \to k \in K$.

Also $b_{n_i} \to b$, and $Tb = k \in K = T(B)$ by the closedness of T, so that $b \in B$ and B is closed.

36) Clearly $\limsup f_n$ is measurable and equals f almost everywhere. Thus f is measurable.

37) (i) and (iii) are straightforward, we show only (ii):
Since h is measurable and $|h| \leq |f|$ we have $0 \leq \int(h^+ + h^-) \leq \int(f^+ + f^-) < \infty$ because f is integrable. It follows that h is integrable.

38) (i) Take R with the Lebesgue measure m. Then R and the empty set and the rationals and the irrationals are two Hahn decompositions for m.

(ii) Let $\{A_1, B_1\}$ and $\{A_2, B_2\}$ be Hahn decompositions for ν. Then $A_1 \backslash A_2$, $A_2 \backslash A_1$, $B_1 \backslash B_2$, and $B_2 \backslash B_1$ are necessarily null sets.

39)(i) Let ν be absolutely continuous with respect to μ and let X be the countable union of the sequence (X_n) with mutually disjoint elements. Let $\nu_{X_n}(E) = \nu(E \cap X_n)$ and define $\mu_{X_n}(E)$ likewise. Then ν_{X_n} is absolutely continuous with respect to μ_{X_n} and by the Radon-Nikodym Theorem we have $\nu_{X_n}(E) = \int_E f_n d\mu_{X_n}$ for some nonnegative measurable functions f_n. Let $f = \sum_{n=1}^{\infty} f_n \chi_{X_n}$. It follows that f is a nonnegative measurable function such that $\nu(E) = \int_E f d\mu$ by the Monotone Convergence Theorem.

40) (i) $X = A \cup B$, $A \cap B = \emptyset$, and $|\nu|B = |\mu|A = 0$. Since ν is absolutely continuous with respect to μ we also have $|\nu|A = 0$. Therefore $|\nu| = 0$, or equivalently $\nu = 0$.