Functional Analysis

July 12, 2007

Contents

1 Metric Spaces

Definition (Metric Space)**.** The pair (X, d), where X is a set and the function

$$d : X \times X \to R$$

is called a metric space if

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ iff $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) \leq d(x, z) + d(z, y)$

Example 1.1 (Metric Spaces)**.**

1. $d(x, y) = |x - y|$ in $\mathbb{R}$.
2. $d(x, y) = [\sum_{i=1}^{n}(x_i - y_i)^2]^{\frac{1}{2}}$ in $\mathbb{R}^n$.
3. $d(x, y) = \|x - y\|$ in a normed space.
4. Let (X, ρ), (Y, σ) be metric spaces and define the Cartesian product $X \times Y = \{(x, y)|x \in X, y \in Y\}$. Then the product measure $\tau((x_1, y_1), (x_2, y_2)) = [\rho(x_1, x_2)^2 + \sigma(y_1, y_2)^2]^{\frac{1}{2}}$.
5. (Subspace) $(Y, \bar{d})$ of (X, d) if $Y \subset X$ and $\bar{d} = d_{|Y \times Y}$.
6. l^∞. Let X be the set of all bounded sequences of complex numbers, i.e., $x = (\xi_i)$ and $|\xi_i| \leq c_x$, $\forall\ i$. Then

   $$d(x, y) = \sup_{i \in N} |\xi_i - \eta_i|$$

   defines a metric on X.
7. $X = C[a, b]$ and

   $$d(x, y) = \max_{t \in [a,b]} |x(t) - y(t)|.$$

8. (Discrete metric)

   $$d(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}.$$

9. l^p. $x = (\xi_i) \in l^p$ if $\sum_{i=1}^{\infty} |\xi_i|^p < \infty$, ($p \geq 1$, fixed),

$$d(x, y) = \left(\sum_{i=1}^{\infty} |\xi_i - \eta_i|^p \right)^{\frac{1}{p}}.$$

Problem 1.

1. Show that $\bar{d}$ is a metric on $C[a, b]$, where

$$\bar{d}(x, y) = \int_a^b |x(t) - y(t)| dt.$$

2. Show that the discrete metric is a metric.

3. Sequence space s: set of all sequences of complex numbers with the metric

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|}. \tag{1}$$

Solution.

1. $\bar{d}(x, y) = 0$ iff $|x(t) - y(t)| = 0$ for all $t \in [a, b]$ because of the continuity. We have $\bar{d}(x, y) \geq 0$ and $\bar{d}(x, y) = \bar{d}(y, x)$ trivially. We can argue the triangle inequality as follows::

$$\bar{d}(x, y) = \int_a^b |x(t) - y(t)| dt \leq \int_a^b |x(t) - z(t)| dt + \int_a^b |z(t) - y(t)| dt = \bar{d}(x, z) + \bar{d}(z, y).$$

2. Left as an exercise.

3. We show only the triangle inequality. Let $a, b \in R$. Then we have the inequalities

$$\frac{|a + b|}{1 + |a + b|} \leq \frac{|a| + |b|}{1 + |a| + |b|} \leq \frac{|a|}{1 + |a|} + \frac{|b|}{1 + |b|},$$

where in the first step we have used the monotonicity of the function

$$f(x) = \frac{x}{1 + x} = 1 - \frac{1}{1 + x}, \text{ for } x > 0.$$

Substituting $a = \xi_i - \zeta_i$ and $b = \zeta_i - \eta_i$, where $x = (\xi_i)$, $y = (\eta_i)$, and $z = (\zeta_i)$ we get

$$\frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|} \leq \frac{|\xi_i - \zeta_i|}{1 + |\xi_i - \zeta_i|} + \frac{|\zeta_i - \eta_i|}{1 + |\zeta_i - \eta_i|}.$$

If we multiply both sides by $\frac{1}{2^i}$ and sum over from $i = 1$ to ∞ we get the stated result.

□

1.1 Open Sets, Closed Sets

Definition (Open Ball, Closed Ball, Sphere)**.**

1. $B(x_0, r) = \{x \in X | d(x, x_0) < r\}$
2. $\bar{B}(x_0, r) = \{x \in X | d(x, x_0) \leq r\}$
3. $S(x_0, r) = \{x \in X | d(x, x_0) = r\}$

Definition (Open, Closed, Interior)**.**

1. M is open if contains a ball about each of its points.
2. $K \subset X$ is closed if $K^c = X - K$ is open.
3. $B(x_0; \varepsilon)$ denotes the ε neighborhood of x_0.
4. $Int(M)$ denotes the interior of M.

Remark 1.2 (Induced Topology)**.** *Consider the set X with the collection τ of all open subsets of X. Then we have*

1. *$\emptyset \in \tau, X \in \tau$.*
2. *The union of any members of τ is a member of τ.*
3. *The finite intersection of members of τ is a member of τ.*

We call the pair (X, τ) a topological space and τ a topology for X. It follows that a metric space is a topological space.

Definition (Continuous)**.** Let $X = (X, d)$ and $Y = (Y, \bar{d})$ be metric spaces. The mapping $T : X \to Y$ is continuous at $x_0 \in X$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\bar{d}(Tx, Tx_0) < \varepsilon \ , \ \forall \ x \text{ such that } d(x, x_0) < \delta.$$

Theorem 1.3 (Continuous Mapping)**.** *$T : X \to Y$ is continuous if and only if the inverse image of any open subset of Y is an open subset of X.*

Proof.

1. Suppose that T is continuous. Let $S \subset Y$ be open S_0 the inverse image of S. Let $S_0 \neq \emptyset$ and take $x_0 \in S_0$. We have $Tx_0 = y_0 \in S$. Since S is open there exists an ε-neighborhood of y_0, say $N \subset S$ such that $y_0 \in N$. The continuity of T implies that x_0 has a δ-neighborhood N_0 which is mapped into N. Since $N \subset S$ we get that $N_0 \subset S_0$, and it follows that S_0 is open.

2. Assume that the inverse image of every open set in Y is an open set in X. Then $\forall\ x_0 \in X$, and N (ε-neighborhood of Tx_0) the inverse image N_0 of N is open. Therefore N_0 contains a δ-neighborhood of x_0. Thus T is continuous.

□

Some more definitions:

Definition (Accumulation Point)**.** $x \in M$ is said to be an accumulation point of M if $\exists\ (x_n) \subset M$ s.t. $x_n \to x$.

Definition (Closure)**.** $\overline{M}$ is the closure of M.

Definition (Dense Set)**.** $M \subset X$ is in X dense if $\overline{M} = X$.

Definition (Separable Space)**.** X is separable if there is a countable subset which is dense in X.

Remark 1.4.

1. *If M is dense, then every ball in X contains a point of M.*
2. *R, C are separable.*
3. *A discrete metric space is separable if and only if it is countable.*

Theorem 1.5. *ℓ^∞ is not separable.*

Proof. Let $y = (\eta_i)$ where $\eta_i = 0, 1$. There are uncountably many y's. If we put small balls with radius $\frac{1}{3}$ at the y's they will not intersect. It follows that if $M \subset l^\infty$ is dense in l^∞, then M is uncountable. Therefore l^∞ is not separable. □

Problem 2. Show that l^p, $1 \leq p < \infty$ is separable.

Solution. Let M the set of all sequences of the form $x = (\xi_1, \xi_2, ..., \xi_n, 0, 0, ...)$, where n is any positive integer and the ξ_is are rational. M is countable. We argue that M is dense in l^p as follows. Let $y = (\eta_i) \in l^p$ be arbitrary. Then for every $\varepsilon > 0$ there is an n such that

$$\sum_{i=n+1}^{\infty} |\eta_i|^p < \frac{\varepsilon^p}{2}.$$

Since the rationals are dense in R, for each η_i there is a rational ξ_i close to it. Hence there is an $x \in M$ such that

$$\sum_{i=1}^{n} |\eta_i - \xi_i| < \frac{\varepsilon^p}{2}.$$

It follows that $d(y, x) < \varepsilon$. □

1.2 Convergence, Cauchy Sequence, Completeness

Definition (Convergent Sequence)**.** We say that the sequence (x_n) is convergent in the metric space $X = (X, d)$ if

$$\lim_{n\to\infty} d(x_n, x) = 0 \text{ for some } x \in X.$$

We shall also use the notation $x_n \to x$.

Definition (Diameter)**.** For a set $M \subset X$ we define the diameter $\delta(M)$ by

$$\delta(M) = \sup_{x,y\in M} d(x, y).$$

M is said to be **bounded** if $\delta(M)$ is finite.

Lemma 1.6.

1. *A convergent sequence (x_n) in X is bounded and its limit x is unique.*
2. *If $x_n \to x$ and $y_n \to y$ in X, then $d(x_n, y_n) \to d(x, y)$.*

Problem 3. Prove Lemma 1.6.

Solution.

1. Suppose that $x_n \to x$. Then, taking $\varepsilon = 1$ we can find an N such that $d(x_n, x) < 1$ for all $n > N$. By the triangle inequality we have

$$d(x_n, x) < 1 + \max d(x_1, x), d(x_2, x), ..., d(x_N, x).$$

Therefore (x_n) is bounded. If $x_n \to x$ and $x_n \to z$, then

$$0 \le d(x, z) \le d(x_n, x) + d(x_n, z) \to 0 \text{ as } n \to \infty$$

and uniqueness of the limit follows.

2. We have

$$d(x_n, y_n) \le d(x_n, x) + d(x, y) + d(y, y_n),$$

and hence

$$d(x_n, y_n) - d(x, y) \le d(x_n, x) + d(y_n, y).$$

Interchanging x_n and x, y_n and y, and multiplying by -1 we get

$$d(x, y) - d(x_n, y_n) \le d(x_n, x) + d(y_n, y).$$

Combining the two inequalities we get

$$|d(x_n, y_n) - d(x, y)| \le d(x_n, x) + d(y_n, y) \to 0 \text{ as } n \to \infty.$$

□

Definition (Cauchy Sequence)**.** (x_n) in (X, d) is a Cauchy sequence if $\forall\ \varepsilon > 0\ \exists\ N = N(\varepsilon)$ s.t.

$$m, n > N \Rightarrow d(x_m, x_n) < \varepsilon.$$

Definition (Complete)**.** X is complete if every Cauchy sequence in X converges in X.

Theorem 1.7. *$\mathbb{R}$, $\mathbb{C}$ are complete metric spaces.*

Theorem 1.8. *Every convergent sequence in a metric space is a Cauchy sequence.*

Theorem 1.9. *Let $M \subset X = (X, d)$, X is a metric space and let $\overline{M}$ denote the closure of M in X. Then*

1. *$x \in \overline{M}$ iff $\exists\ (x_n) \in M$ s.t. $x_n \to x$.*
2. *M is closed iff $x_n \in M$ and $x_n \to x$ imply that $x \in M$.*

Theorem 1.10. *Let X be a complete metric space and $M \subset X$. M is complete iff M is closed in X.*

Theorem 1.11. *Let T be a mapping from (X, d) into $(Y, \tilde{d})$. $T : X \to Y$ is continuous at $x_0 \in X$ iff $x_n \to x_0$ implies $Tx_n \to Tx_0$.*

Problem 4.

1. Prove Theorem 1.10
2. Prove Theorem 1.11

Solution.

1. Let M be complete. Then for every $x \in \overline{M}$ there is a sequence (x_n) which converges to x. Since (x_n) is Cauchy and M is complete $x_n \to x \in M$, therefore M is closed.

   Conversely, let M be closed and (x_n) be Cauchy in M. Then $x_n \to x \in X$, which implies $x \in \overline{M}$, and therefore $x \in M$. Thus M is complete.

2. Assume that T is continuous at x_0. Then for $\varepsilon > 0\ \exists\ \delta > 0$ such that

   $$d(x, x_0) < \delta \text{ implies } \tilde{d}(Tx, Tx_0) < \varepsilon.$$

   Take a sequence (x_n) such that $x_n \to x_0$. Then $\exists\ N$ s.t. $\forall\ n > N$ we have $d(x_n, x_0) < \delta$ and hence

   $$\forall\ n > N,\ \tilde{d}(Tx_n, Tx)) < \varepsilon.$$

   Conversely, assume that $x_n \to x) \Rightarrow Tx_n \to Tx_0$ and show that T is continuous at x_0. Otherwise, $\exists\ \varepsilon > 0$ such that $\forall\ \delta > 0$, $\exists\ x \neq x_0)$ satisfying $d(x, x_0) < \delta$ but $\tilde{d}(Tx, Tx_0) \geq \varepsilon$. Let $\delta = \frac{1}{n}$. Then there is an x_n such that $d(x_n, x_0) < \frac{1}{n}$ but $\tilde{d}(Tx_n, Tx_0) \geq \varepsilon$ contrary to our assumption.

□

1.3 Completeness Proofs

Theorem 1.12. $\mathbb{R}^n, \mathbb{C}^n$ *are complete.*

Theorem 1.13. ℓ^∞ *is complete.*

Proof. Let (x_m) be a Cauchy sequence in ℓ^∞.

$$\Rightarrow d(x_m, x_n) = \sup_i |x_i^{(m)} - x_i^{(n)}| < \varepsilon, \text{ for } m, n > N(\varepsilon).$$

$\Rightarrow$ for fixed i, the sequence $(\xi_i^{(m)})$ is Cauchy in R, and therefore we have that $\xi_i^{(m)} \to \xi_i$, for $i = 1, 2, 3, \ldots$ Let $x = (\xi_i)$. It follows easily that $x \in \ell^\infty$ and $x_m \to x$. □

Theorem 1.14. *The space of convergent sequences $x = (\xi_i)$ of complex numbers with the metric induced from ℓ^∞ is complete.*

Theorem 1.15. ℓ^p *is complete if* $1 \leq p < \infty$.

Theorem 1.16. $C[a, b]$ *is complete.*

Proof. Let (x_m) be a Cauchy sequence in $C[a, b]$. Then

$$\forall\, \varepsilon > 0\ \exists\, N \text{ s.t. for } m, n > N \quad \Rightarrow \quad d(x_m, x_n) = \max_{t \in [a,b]} |x_m(t) - x_n(t)| < \varepsilon. \qquad (2)$$

$\Rightarrow$ for any fixed $t_0 \in [a, b]$ we have that $|x_m(t_0) - x_n(t_0)| < \varepsilon$. It follows that $x_m(t_0)$ is Cauchy and therefore $x_m(t_0) \to x(t_0)$. Show that $x(t) \in C[a, b]$ and $x_m \to x$. From (2) with $n \to \infty$ we get

$$\max_{t \in [a,b]} |x_m(t) - x(t)| \leq \varepsilon.$$

Therefore $x_m(t)$ converges uniformly to $x(t)$ on $[a, b]$. Since the x_m's are continuous on $[a, b]$ and the convergence is uniform $x(t)$ is continuous, and thus belongs to $C[a, b]$. □

Remark 1.17. *Note that in $C[a, b]$ convergence is uniform convergence; we also use the terminology uniform metric for the metric generated by the "sup"-norm.*

Example 1.18 (Incomplete Metric Spaces)**.**

1. $\mathbb{Q}$
2. The polynomials
3. $C[a, b]$ with the $\|\cdot\|_2$ norm

1.4 Completion of Metric Spaces

Definition (Isometry)**.** Let $X = (X, d)$ and $\bar{X} = (\bar{X}, \bar{d})$ be metric spaces. The mapping $T : X \to \bar{X}$ is an isometry if $\bar{d}(Tx, Ty) = d(x, y)$. The metric spaces X and $\bar{X}$ are isometric if there is a bijective isometry of X onto $\bar{X}$

Theorem 1.19. *Let $X = (X, d)$ be a metric space. Then there exists a complete metric space $\hat{X} = (\hat{X}, \hat{d})$ which has a subspace W that is isometric with X and is dense in $\hat{X}$. $\hat{X}$ is unique except for isometries.*

Proof. This proof is divided into steps.

1. Construction of $\hat{X}$. Let (x_n) and $(x_n^{'})$ be equivalent Cauchy sequences, i.e.,

$$\lim_{n\to\infty} d(x_n, x_n^{'}) = 0.$$

Define $\hat{X}$ to be the set of all equivalence classes $\hat{x}$ of Cauchy sequences. Define now

$$\hat{d}(\hat{x}, \hat{y}) = \lim_{n\to\infty} d(x_n, y_n) \text{ where } (x_n) \in \hat{x} \text{ and } (y_n) \in \hat{y}. \tag{3}$$

We show that limit in (3) exists and independent of the particular choice of the representatives. We have using the triangle inequality

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n).$$

It follows that

$$d(x_n, y_n) - d(x_m, y_m) \leq d(x_n, x_m) + d(y_m, y_n).$$

Exchanging the role of n and m we get

$$d(x_m, y_m) - d(x_n, y_n) \leq d(x_n, x_m) + d(y_m, y_n).$$

The two inequality together yield

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_m, y_n).$$

It follows that $(d(x_n, y_n))$ is a Cauchy sequence in R and therefore it converges, i.e., the limit in (3) exists.

We show now that the limit in (3) is independent of the particular choice of representatives. Let $(x_n), (x_n^{'}) \in \hat{x}$ and $(y_n), (y_n^{'}) \in \hat{y}$. Then

$$|d(x_n, y_n) - d(x_n^{'}, y_n^{'})| \leq d(x_n, x_n^{'}) + d(y_n, y_n^{'}) \to 0 \text{ as } n \to \infty,$$

which implies

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x_n^{'}, y_n^{'}).$$

2. Show $\hat{d}$ is a metric on $\hat{X}$. Left as an exercise.

3. Construction of an isometry $T : X \to W \in \hat{X}$. With each $b \in X$ we associate the class $\hat{b} \in \hat{X}$ which contains the constant Cauchy sequence $(b, b, b, ...)$. This defines the mapping $T : X \to W$ onto the subspace $W = T(X) \in \hat{X}$. The mapping T is given by $b \to \hat{b} = Tb$, where $(b, b, ...) \in \hat{b}$. According to (3) $\hat{d}(\hat{b}, \hat{c}) = d(b, c)$, so T is an isometry. T is onto W since $T(X) = W$. (W and X are isometric.)

   Show that W is dense in $\hat{X}$. Let $\hat{x} \in \hat{X}$ be arbitrary and let $(x_n) \in \hat{x}$. For every $\varepsilon > 0$ there is an N such that $d(x_n, x_N) < \frac{\varepsilon}{2}$, for $n > N$. Let $(x_N, x_N, ...) \in \hat{x}_N$. Then $\hat{x}_N \in W$, and by (3)

$$\hat{d}(\hat{x}, \hat{x}_N) = \lim_{n\to\infty} d(x_n, x_N) \leq \frac{\varepsilon}{2} < \varepsilon.$$

4. Completeness of $\hat{X}$. Let $(\hat{x}_n)$ be an arbitrary Cauchy sequence in $\hat{X}$. Since W is dense in $\hat{X}$ for every $\hat{x}_n$ there is a $\hat{z}_n \in W$ such that

$$\hat{d}(\hat{x}_n, \hat{z}_n) < \frac{1}{n}.$$

   It follows using the triangle inequality and the Cauchyness of $(\hat{x}_n)$ that $(\hat{z}_n)$ is Cauchy. Then (z_n), where $z_n = T^{-1}\hat{z}_n$ is Cauchy in X. Let $\hat{x} \in \hat{X}$ be the class to which (z_n) belongs. We have

$$\hat{d}(\hat{x}_n, \hat{x}) \leq \hat{d}(\hat{x}_n, \hat{z}_n) + \hat{d}(\hat{z}_n, \hat{x}) < \frac{1}{n} + \hat{d}(\hat{z}_n, \hat{x}) < \varepsilon$$

   for sufficiently large n.

5. Uniqueness of $\hat{X}$. If $(\bar{X}, \bar{d})$ is another complete metric space with a subspace $\bar{W}$ dense in $\bar{X}$ and isometric with X, then $\bar{W}$ is isometric with W and the distances on $\bar{X}$ and $\hat{X}$ must be the same. Hence $\bar{X}$ and $\hat{X}$ are isometric.

□

Problem 5. Show now that $\hat{d}$ is a metric on $\hat{X}$.

Solution. Clearly, $\hat{d}(\hat{x}, \hat{y}) \geq 0$ because of the definition (3). Also, $\hat{d}(\hat{x}, \hat{x}) = 0$, and $\hat{d}(\hat{x}, \hat{y}) = \hat{d}(\hat{y}, \hat{x})$, because of the properties of d.

$$\hat{d}(\hat{x}, \hat{y}) \leq \hat{d}(\hat{x}, \hat{z}) + \hat{d}(\hat{z}, \hat{y})$$

because we have

$$d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n) \text{ for all } n.$$

□

Problem 6. A homeomorphism is a continuous bijective mapping $T : X \to Y$ whose inverse is continuous. If such mapping exists, then X and Y are homeomorphic.

1. Show that is X and Y are isometric, then they are homeomorphic.

2. Illustrate with an example that a complete and an incomplete metric space may be homeomorphic.

Solution. 1. If $f : (X, d) \to (Y, \hat{d})$ is an isometry, then $\hat{d}(f(x), f(y)) = d(x, y) < \epsilon$ whenever $d(x, y) < \delta$, so f is continuous. If (X, d) and $(Y, \hat{d})$ are isometric, then there exists a bijective isometry $f : X \to Y$. It follows that $f^{-1} : Y \to X$ is also an isometry, and that both f and f^{-1} are continuous. Therefore X and Y are homeomorphic.

2. Let $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to R$ be given by $f(x) = \tan x$. f is cintinuous and bijective. Moreover, $f^{-1} : R \to (-\frac{\pi}{2}, \frac{\pi}{2})$ given by $f^{-1}(x) = \tan^{-1}(x)$ is also continuous. So, R is homeomorphic to $(-\frac{\pi}{2}, \frac{\pi}{2})$.

□

Problem 7. If (X, d) is complete, show that $(X, \bar{d})$, where $\bar{d} = \frac{d}{1+d}$ is complete.

Solution. It is easy to see that if (x_n) is Cauchy in $(X, \bar{d})$, then (x_n) is Cauchy in (X, d). Since (X, d) is complete, $x_n \to x$, but then we also have $x_n \to x$ in $(X, \bar{d})$. It follows that $(X, \bar{d})$ is complete. □

2 Normed Spaces

Vector spaces, linear independence, finite- and infinite-dimensional vector spaces.

Definition (Basis)**.** If X is any vector space, and B is a linearly independent subset of X which spans X, then B is called a basis (or Hamel basis) for X.

Definition (Banach Space)**.** A complete normed space.

Definition (norm)**.** A real-valued function $\|\cdot\| : X \to \mathbb{R}$ satisfying

1. $\|x\| \geq 0$
2. $\|x\| = 0$ iff $x = 0$
3. $\|\alpha x\| = |\alpha|\|x\|$
4. $|x + y| \leq |x| + |y|$

Definition (Induced Metric)**.** In a normed space (X, d) we can always define a metric by $d(x, y) = \|x - y\|$.

Remark 2.1. *All previous results about metric spaces apply to normed spaces with the induced metric.*

Problem 8. The norm is continuous, that is $\|\cdot\| : X \to \mathbb{R}$ by $x \mapsto \|x\|$ is a continuous.

Solution. The triangle inequality implies that

$$|\|x\| - \|y\|| \leq \|x - y\|.$$

□

Example 2.2 (Norm Spaces)**.** $\mathbb{R}^n, \mathbb{C}^n, \ell^p, \ell^\infty, C[a, b], \mathcal{L}^2[a, b], \mathcal{L}^p[a, b]$.

Remark 2.3 (Induced Norm is Translation Invariant)**.** *A metric induced by a norm on a normed space is translation invariant, i.e.,* $d(x + a, y + a) = d(x, y)$ *and* $d(\alpha x, \alpha y) = |\alpha|d(x, y)$*. For example the metric (1) is not coming from a norm.*

Theorem 2.4. *A subspace Y of a Banach space X is complete iff the set Y is closed in X.*

Definition (Convergent, Cauchy, Absolutely Convergent)**.**

1. (x_n) is convergent if $\|x_n - x\| \to 0$. (x is the limit of (x_n)).
2. (x_n) is Cauchy if $\|x_m - x_n\| < \epsilon$ for $m, n > N(\epsilon)$.

Definition (Infinite series)**.** Let (x_k) be a sequence and consider the partial sums

$$s_n = x_1 + x_2 + ... + x_n.$$

If s_n converges, say $s_n \to s$, then $\sum_{k=1}^{\infty} x_k$ is convergent and

$$s = \sum_{k=1}^{\infty}.$$

If $\|x_1\| + \|x_2\| + ... + \|x_n\| + ...$ converges, then $\sum_{k=1}^{\infty} x_k$ is absolutely convergent.

Remark 2.5. *Absolute convergence implies convergence iff X is complete.*

Definition (Schauder basis)**.** If X contains a sequence (e_n) such that for every $x \in X$ there exists a unique sequence (α_n) with the property that

$$\|x - (\alpha_1 e_1 + ... + \alpha_n e_n)\| \to 0 \text{ as } n \to \infty,$$

then (e_n) is called a Schauder Basis for X. The representation

$$x = \sum_{k=1}^{\infty} \alpha_k e_k$$

is called the expansion of x with respect to (e_n).

Remark 2.6. *If X has a Schauder basis, then X is separable.*

Theorem 2.7. *Let $X = (X, \|\cdot\|)$ be a normed space. Then there is a Banach space $\hat{X}$ and an isometry A from X onto a subspace $W \subset \hat{X}$ which is dense in $\hat{X}$. $\hat{X}$ is unique, except for isometries.*

Theorem 2.8. *Let X be a normed space, where the absolute convergence of any series always implies convergence. Then X is complete.*

Proof. Let (s_n) be any Cauchy sequence in X. Then for every $k \in N$ there exists n_k such that $\|s_n - s_m\| < 2^{-k}$, $(m, n > n_k)$ and we can choose $n_{k+1} > n_k$ for all k. Then (s_{n_k}) is a subsequence of (s_n) and is the sequence of the partial sums of $\sum x_k$, where $x_1 = s_{n_1}, ..., x_k = s_{n_k} - s_{n_{k-1}}, ...$. Hence

$$\sum \|x_k\| \leq \|x_1\| + \|x_2\| + \sum 2^{-k} = \|x_1\| + \|x_2\| + 1.$$

It follows that $\sum x_k$ is absolutely convergent. By assumption, $\sum x_k$ converges, say, $s_{n_k} \to s \in X$. Since (s_n) is Cauchy, $s_n \to s$ because

$$\|s_n - s\| \leq \|s_n - s_{n_k}\| + \|s_{n_k} - s\|.$$

Since (s_n) was arbitrary, $s_n \to s$ shows that X is complete. □

2.1 Finite dimensional Normed Spaces or Subspaces

Theorem 2.9 (Linear Combinations)**.** *Let X be a normed space, and let*

$$\{x_1, \dots, x_n\}$$

be a linearly independent set of vectors in X. Then $\exists\ c > 0$ s.t.

$$\|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \geq c\left(|\alpha_1| + \cdots + |\alpha_n|\right).$$

Proof. Let

$$s = |\alpha_1| + \cdots + |\alpha_n|.$$

If $s = 0$ we are done, so assume $s > 0$. Define

$$\beta_i = \frac{\alpha_i}{s}.$$

Clearly we have that

$$\sum_{i=1}^{n} |\beta_i| = 1.$$

Then we can reduce the above inequality to

$$\left\| \sum_{i=1}^{n} \beta_i x_i \right\| \geq c. \tag{4}$$

We will prove the result for Theorem 4.

Suppose Theorem 4 is not true. Then $\exists\ (y_m)$

$$y_m = \sum_{i=1}^{n} \beta_i^{(m)} x_i$$

such that

$$\|y_m\| \to 0 \text{ as } m \to \infty.$$

Note that $\sum_{i=1}^{n} |\beta_i^{(m)}| = 1 \Rightarrow |\beta_i^{(m)}| \leq 1\ \forall\ i$. Hence for each **fixed** i the sequence

$$\left(\beta_i^{(m)}\right) = \left(\beta_i^{(1)}, \beta_i^{(2)}, \dots\right)$$

is bounded. Therefore $\beta_1^{(m)}$ has a convergent subsequence (B-W). Let β_1 be the limit, and let $(y_{1,m})$ be the corresponding subsequence of (y_m)

$$(y_{1,m}) = \gamma_1^{(m)} x_1 + \sum_{i=2}^{n} \beta_i^{(m)} x_i$$

$$\gamma_1^{(m)} \to \beta_1.$$

Then there is a there exist corresponding $\beta_2, (y_{2,m})$ where

$$\beta_2^{(m)} \to \beta_2 \text{ as } n \to \infty$$

$$(y_{2,m}) = \gamma_1^{(m)} x_1 + \gamma_2^{(m)} x_2 + \sum_{i=3}^{n} \beta_i^{(m)} x_i.$$

Since n is finite this process will terminate with

$$(y_{n,m}) = \sum_{i=1}^{n} \gamma_i^{(m)} x_i$$

with $(y_{n,m})$ a subsequence of (y_m). By construction,

$$\gamma_i^{(m)} \to \beta_i \ \forall \ i$$

$$\sum_{i=1}^{n} |\beta_i| = 1.$$

Therefore $(y_{n,m})$ has a limit, say

$$y = \sum_{i=1}^{n} \beta_i x_i.$$

Since $\|y_m\| \to 0$ by assumption we must also have $\|y_{n,m}\| \to 0$, We know $y \neq 0$ since $\sum_{i=1}^{n} |\beta_i| = 1$ and $\{x_i\}$ is a basis, a contradiction. □

Theorem 2.10 (Completeness)**.** *Every f.d.s.s. Y of a n.s. X is complete. In particular, every f.d.n.s. is complete.*

Proof. Let (y_m) be a Cauchy sequence in Y. We must find a limit y such that

$$y_m \to y$$

$$y \in Y.$$

Let $n = \dim Y$, and let $\{e_1, \ldots, e_n\}$ be a basis for Y. Then $\forall \ m$ we can represent y_m as

$$y_m = \sum_{i=1}^{n} \alpha_i^{(m)} e_i.$$

But then for each **fixed** i sequence $(\alpha_i^{(m)})$ is Cauchy in $\mathbb{R}$. So each of these sequence converges, say

$$\alpha_i^{(m)} \to \alpha_i.$$

Then define

$$y = \sum_{i=1}^{n} \alpha_i e_i.$$

Clearly, $y \in Y$ and $y_m \to y$.

$\square$

Theorem 2.11. *Every f.d.s.s. Y of a closed n.s. X is closed in X.*

Proof. By Theorem 2.10 Y must be complete. Therefore Y is closed in X. $\square$

Note that finiteness in the previous proof is essential. Infinite dimensional subspaces need not be closed in X. For example consider

$$X = C[0,1]$$

$$Y = \text{span}\left(\{1, t, ..., t^n, ..\}\right).$$

Then the sequence (t^i) converges to

$$f(t) = \begin{cases} 0 & t < 1 \\ 1 & t = 1 \end{cases}$$

which is not in Y.

Definition (Equivalent Norms)**.** $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if $\exists\ a, b > 0$ s.t.

$$a\,\|x\|_2 \leq \|x\|_1 \leq b\,\|x\|_2\,,\ \forall\ x \in X.$$

Theorem 2.12 (Finite Dimesional $\Rightarrow$ All Norms are Equivalent)**.** *On a f.d.n.s. X every two norms $\|\cdot\|_1$, $\|\cdot\|_2$ are equivalent.*

Proof. Let $n = \dim X$, let $\{e_1, \ldots, e_n\}$ be a basis for X with $\|e_i\|_1 = 1$, let $x \in X$ be represented as

$$x = \sum_{i=1}^{n} \alpha_i e_i.$$

Then $\exists\ c > 0$ s.t.

$$\|x\|_1 \geq c \sum_{i=1}^{n} |\alpha_i|.$$

Also note

$$\|x\|_2 \leq \sum_{i=1}^{n} |\alpha_i|\, \|e_i\|_2 \leq k \sum_{i=1}^{n} |\alpha_i|$$

where

$$k = \max\left\{\|e_i\|_2\right\}.$$

Combining these inequalities we obtain

$$\|x\|_2 \leq \frac{k}{c} c \sum_{i=1}^{n} |\alpha_i| \leq \frac{1}{a} \|x\|_1 ,$$

where

$$a = \frac{c}{k}$$

and we obtain the inequality involving a. A similar argument yields for the inequality involving b. □

Problem 9. What is the largest possible $c > 0$ in

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \geq c \sum_{i=1}^{n} |\alpha_i|$$

if

1. $X = \mathbb{R}^2$, $x_1 = (1, 0)$, $x_2 = (0, 1)$.
2. $X = \mathbb{R}^3$, $x_1 = (1, 0, 0)$, $x_2 = (0, 1, 0)$, $x_3 = (0, 0, 1)$.

Solution. Find $\min\{\| \sum \beta_i x_i \| : \sum |\beta_i| = 1\}$ and obtain $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{3}}$, respectively. □

Problem 10. Show that if $\|\cdot\|_1, \|\cdot\|_2$ are equivalent norms on X then the Cauchy sequences in $(X, \|\cdot\|_1), (X, \|\cdot\|_2)$ are the same.

Solution. Suppose that (x_n) is a Cauchy sequence in $\| \cdot \|_2$. Then for every $\epsilon > 0$ there exists N such that $\|x_n - x_m\|_2 < \frac{\epsilon}{b}$ for all $m, n \geq N$, where b is a constant for which $\|x\|_1 \leq b\|x\|_2$. Then $\|x_n - x_m\|_1 \leq b\|x_n - x_m\|_2 < \epsilon$ for all $m, n \geq N$. It follows that (x_n) is Cauchy in $\| \cdot \|_1$. A similar argument works in the other direction. □

2.2 Compactness and Finite Dimension

Definition (Compactness)**.** A metric space (X, d) is compact if every sequence in X has a convergent subsequence. $M \subset X$ is compact if M is compact as a subspace of X, i.e., if every sequence in M has a convergent subsequence which converges to an element in M.

Theorem 2.13. *If M is compact then M is closed and bounded.*

Proof. First we show closed. For all $x \in \overline{M}$ $\exists$ $(x_n) \subset M$ s.t. $x_n \to x$. M is compact $\Rightarrow$ $x \in M$, hence M is closed.

To show boundedness assume it is not true. Then fix $b \in M$ and choose $y_n \in M$ s.t. $d(y_n, b) > n$. Then $(y_n) \subset M$ is a sequence with no convergent subsequence, contradicting the assumption that M is compact. Thus M is bounded. □

Note the converse of this theorem is false. Consider $M \subset \ell^2$ as

$$M = \left\{ x_n \in \ell^2 \text{ s.t. } x_n = (e_n), (e_n)_i = \delta_{ni} \right\}.$$

Then M is bounded since $\|x_n\| = 1$ and M is closed because there are no accumulation points

$$\|x_n - x_m\| = \sqrt{2}.$$

But the sequence (x_n) has no convergent subsequence, so M is not compact. In some sense, there is too much room in the infinite dimensional setting.

Theorem 2.14 (Compactness)**.** *Let X be a f.d.n.s. . Then $M \subset X$ is (sequentially) compact iff M is closed and bounded.*

Proof. The "⇒" case was proved by the previous theorem. For the other direction, assume M is closed and bounded.

For the other direction ("⇐"), assume $n = \dim X, \{e_1, \ldots, e_n\}$ is a basis for X, and $(x_m) \subset M$ is an arbitrary sequence. Then there is a representation

$$x_m = \sum_{i=1}^{m} \xi_i^{(m)} e_i, \ \forall\ m.$$

Since M is bounded, (x_m) is bounded so $\exists\ k \geq 0$ s.t.

$$\|x_m\| \leq k.$$

So then

$$\begin{aligned} k &\geq \|x_m\| \\ &= \left\| \sum_{i=1}^{n} \xi_i^{(m)} e_i \right\| \\ &\geq c \sum_{i=1}^{n} |\xi_i^{(m)}| \end{aligned}$$

and therefore $\forall\ i, (\xi_i^{(m)}) \subset \mathbb{R}$ is bounded and therefore $\forall\ i, (\xi_i^{(m)})$ has an accumulation point z. Since M is closed, $z \in M$ and there is a subsequence $(z_m) \subset (x_m)$ s.t. $z_m \to z$, say,

$$z = \sum_{i=1}^{n} \xi_i e_i.$$

Since (x_m) was arbitrary, M is compact. □

We shall need the following technical result to prove subsequent theorems.

Theorem 2.15 (Riesz)**.** *Let Y, Z be subspaces of X, and suppose Y is closed and is a proper subspace of Z. Then $\forall\ \theta \in (0,1)\ \exists\ z$ s.t.*

$$\|z\| = 1$$

$$\|z - y\| \geq \theta,\ \forall\ y \in Y.$$

Proof. Let $v \in Z - Y$ be arbitrary and let

$$a = \inf_{y \in Y} \|v - y\|.$$

Note if $a = 0$ then $v \in Y$ since Y is closed. Therefore $a > 0$.

Choose $\theta \in (0,1)$. Then $\exists\ y_0 \in Y$ s.t.

$$a \leq \|v - y_0\| \leq \frac{a}{\theta}.$$

Let

$$z = c(v - y_0) \text{ where } c = \frac{1}{\|v - y_0\|}.$$

By construction, $\|z\| = 1$. We want to show $\|z - y\| \geq \theta,\ \forall\ y \in Y$. So

$$\begin{aligned} \|z - y\| &= \|c(v - y_0) - y\| \\ &= c\left\|v - y_0 - \frac{y}{c}\right\| \\ &= c\,\|v - y_1\| \end{aligned}$$

for some $y_1 \in Y, \Rightarrow \|v - y_1\| \geq a \Rightarrow$

$$\begin{aligned} \|z - y\| &= \|v - y_1\| \\ &\geq ca \\ &= \frac{a}{\|v - y_0\|} \\ &\geq \frac{a}{\frac{a}{\theta}} \\ &= \theta. \end{aligned}$$

□

Theorem 2.16 (Finite Dimension and Compactness)**.** *Let X be a normed space. Then the closed unit ball*

$$M = \{x \text{ s.t. } \|x\| \leq 1\}$$

is compact iff $\dim X < \infty$.

Proof. "$\Rightarrow$". Suppose M is compact, but $\dim X = \infty$. Pick $x_1, \|x_1\| = 1$. Then x_1 generates a closed, 1-dimensional subspace $X_1 \subsetneqq X$. Then also $\exists\, x_2 \in X$ s.t. $\|x_2\| = 1$ and

$$\|x_2 - x_1\| \geq \theta = \frac{1}{2}.$$

Then x_1, x_2 generate a closed, 2-dimensional subspace $X_2 \subsetneqq X$. Then $\exists\, x_3 \in X$ s.t. $\|x_3\| = 1$ and

$$\|x_3 - x_i\| \geq \theta = \frac{1}{2},\ \forall\, i = 1, 2.$$

In this way construct x_n s.t.

$$\|x_n - x_i\| \geq \theta = \frac{1}{2},\ \forall\, i = 1, \ldots, n-1.$$

Then $(x_n) \subset X$ is a sequence such that $\|x_m - x_n\| \geq \frac{1}{2},\ \forall\, m, n$. Therefore (x_n) has no accumulation point, contradicting that M was compact.

"$\Leftarrow$" was proved by Theorem 2.14. □

Theorem 2.17 (Continuous Preserves Compact)**.** *Let $T : X \to Y$ be continuous. Then $M \subset X$ compact $\Rightarrow T(M) \subset Y$ compact.*

Proof. Let $(y_n) \subset T(M)$ be arbitrary. Then $\forall\, n\ \exists\, x_n$ s.t. $y_n = Tx_n$. Then $(x_n) \subset M$ has a convergent subsequence (x_{n_k}). Since T is continuous, the image $(y_{n_k}) = T(x_{n_k})$ also converges. Therefore $T(M)$ is compact. □

Corollary 2.18 (Max, Min)**.** *Let $T : X \to \mathbb{R}$ be continuous and let $M \subset X$ be compact. Then T obtains its max and min on M.*

Proof. $T(M)$ is compact $\Rightarrow T(M)$ is closed and bounded. Therefore

$$\inf T(M) \in T(M)$$

$$\sup T(M) \in T(M).$$

□

Problem 11. Show that a discrete metric space with infinitely many points is not compact.

Solution. The space contains an infinite sequence (x_n), $x_n \neq x_m$ $(n \neq m)$ which cannot have a convergent subsequence since $d(x_n, x_m) = 1$. □

Problem 12 (Local Compactness). A metric space X is said to be locally compact if every point $x \in X$ has a compact neighborhood. Show that $\mathbb{R}, \mathbb{R}^n, \mathbb{C}, \mathbb{C}^n$ are locally compact.

Solution. The closed ball $\bar{B}(x, \epsilon)$ around an arbitrary point x is compact. □

2.3 Linear Operators

Definition (Linear, Domain and Range)**.** Let X, Y be vector spaces and let $T : X \overset{\text{into}}{\longrightarrow} Y$. Then

1. If $\forall\, x, y \in \mathcal{D}(T)$ and scalars α, β

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty$$

   then T is said to be a linear operator.

2. The null space of T is $\mathcal{N}(T) = \{x \in \mathcal{D}(T) \text{ s.t. } Tx = 0\}$

3. The domain of T, $\mathcal{D}(T)$, is a vector space

4. The range of T, $\mathcal{R}(T)$, lies in a vector space

Note that obviously $\mathcal{D}(T) \subset X$. It is customary to write

$$T : \mathcal{D}(T) \overset{\text{onto}}{\longrightarrow} \mathcal{R}(T).$$

Also, $T0 = T(0+0) = T0 + T0 \Rightarrow T0 = 0 \Rightarrow \mathcal{N}(T) \neq \emptyset$.

Example 2.19 (Differentiation)**.** Let X be the space of polynomials on $[a, b]$. Then define $T : X \overset{\text{onto}}{\longrightarrow} X$

$$Tx(t) = \frac{d}{dt}x(t).$$

Example 2.20 (Integration)**.** Let $X = C[a, b]$. Then define $T : X \overset{\text{into}}{\longrightarrow} X$ by

$$Tx(t) = \int_a^t x(s)ds.$$

Example 2.21 (Multiplication)**.** Let $X = C[a, b]$. Then define $T : X \overset{\text{into}}{\longrightarrow} X$ by

$$Tx(t) = tx(t).$$

Example 2.22 (Matrices)**.** Let $A = (a_{ij})$. Then define $T : \mathbb{R}^m \to \mathbb{R}^n$ by

$$Tx = Ax.$$

Theorem 2.23 (Range and Null Space)**.** *Let T be a linear operator.*

1. *$\mathcal{R}(T)$ is a vector space.*

2. *If $\dim \mathcal{D}(T) = n < \infty$ then $\dim \mathcal{R}(T) \leq n$.*

3. $\mathcal{N}(T)$ is a vector space.

Proof. 1. Let $y_1, y_2 \in \mathcal{R}(T)$ and let α, β be scalars. Since $y_1, y_2 \in \mathcal{R}(T)$, $\exists\ x_1, x_2 \in \mathcal{D}(T)$ s.t.

$$Tx_1 = y_1$$

$$Tx_2 = y_2.$$

Since $\mathcal{D}(T)$ is vector space, $\alpha x_1 + \beta x_2 \in \mathcal{D}(T)$, so

$$T(\alpha x_1 + \beta x_2) \in \mathcal{R}(T).$$

But

$$T(\alpha x_1 + \beta x_2) = \alpha Tx_1 + \beta Tx_2 = \alpha y_1 + \beta y_2$$

means $\mathcal{R}(T)$ is a vector space.

2. Pick $n+1$ elements arbitrarily in $\mathcal{R}(T)$. Then we have

$$y_i = Tx_i,\ \forall\ i = 1, \ldots, n+1$$

for some $\{x_1, \ldots, x_{n+1}\}$ in $\mathcal{D}(T)$. Since $\dim \mathcal{D}(T) = n$, this set must be linearly dependent. Similarly, the points in the range are also linearly dependent. Therefore, $\dim \mathcal{R}(T) \leq n$.

3. Let $x_1, x_2 \in \mathcal{N}(T)$, i.e. ,$Tx_1 = Tx_2 = 0$, and let α, β be scalars. But then

$$T(\alpha x_1 + \beta x_2) = \alpha Tx_1 + \beta Tx_2 = 0 \Rightarrow \alpha x_1 + \beta x_2 \in \mathcal{N}(T).$$

Therefore $\mathcal{N}(T)$ is a vector space.

□

The next result shows linear operators preserve linear independence.

Definition (Injective (One-to-one))**.** $T : \mathcal{D}(T) \xrightarrow{\text{into}} Y$ is injective (or one-to-one) if $\forall\ x_1, x_2 \in \mathcal{D}(T)$ s.t. $x_1 \neq x_2$

$$Tx_1 \neq Tx_2.$$

Equivalently,

$$Tx_1 = Tx_2 \Rightarrow x_1 = x_2.$$

So therefore if $T : \mathcal{D}(T) \xrightarrow{\text{into}} Y$ is injective then $\exists\ T^{-1} : \mathcal{R}(T) \xrightarrow{\text{onto}} \mathcal{D}(T)$

$$y_0 \mapsto x_0$$

$$y_0 = Tx_0$$

and therefore

$$T^{-1}Tx = x,\ \forall\ x \in \mathcal{D}(T)$$

$$TT^{-1}y = y,\ \forall\ y \in \mathcal{R}(T).$$

Theorem 2.24 (Inverses). *Let X, Y be vector spaces and $T : \mathcal{D}(T) \xrightarrow{into} Y$ be linear with*

$$\mathcal{D}(T) \subset X \text{ and } \mathcal{R}(T) \subset Y.$$

1. *$T^{-1} : \mathcal{R}(T) \xrightarrow{onto} \mathcal{D}(T)$ exists iff $Tx = 0 \Rightarrow x = 0$.*
2. *If T^{-1} exists then it is linear.*
3. *If $\dim \mathcal{D}(T) = n < \infty$ and T^{-1} exists then $\dim \mathcal{R}(T) = \dim \mathcal{D}(T)$.*

Proof.

1. Suppose $Tx = 0 \Rightarrow x = 0$. Then

$$Tx_1 = Tx_2 \Rightarrow T(x_1 - x_2) = 0 \Rightarrow x_1 = x_2$$

   So T is injective T^{-1} exists.

   Conversely, if T^{-1} exists then $Tx_1 = Tx_2 \Rightarrow x_1 = x_2$. Therefore

$$Tx = T0 = 0 \Rightarrow x = 0.$$

2. We assume that T^{-1} exists and show that it is linear in a straightforward fashion.
3. It follows from the earlier result applied to both T and T^{-1}.

□

Theorem 2.25 (Inverse of Product (Composition)). *Let $T : X \to Y$ and $S : Y \to Z$ be bijections. Then $(ST)^{-1} : Z \to X$ exists and*

$$(ST)^{-1} = T^{-1}S^{-1}.$$

2.4 Bounded and Continuous Linear Operators

Definition (Bounded Linear Operator). Let X, Y be normed spaces and $T : \mathcal{D}(T) \xrightarrow{\text{into}} Y, \mathcal{D}(T) \subset X$. Then T is said to be bounded if $\exists\, k > 0$ s.t.

$$\|Tx\| \le k \|x\|, \ \forall\, x \in \mathcal{D}(T).$$

Definition (Induced Operator Norm). Let X, Y be normed spaces and $T : \mathcal{D}(T) \xrightarrow{\text{into}} Y, \mathcal{D}(T) \subset X$. Then the norms on X, Y induce a norm for operators

$$\|T\| = \sup_{\substack{x \in \mathcal{D}(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}.$$

Equivalently,

$$\|T\| = \sup_{\substack{x \in \mathcal{D}(T) \\ \|x\| = 1}} \|Tx\|.$$

The preceding definition implies that

$$\|Tx\| \leq \|T\| \, \|x\| \, , \ \forall \ x.$$

Example 2.26 (Differention is Unbounded)**.** Let $T : C[0,1] \rightarrow C[0,1]$,be defined by $Tx(t) = x^{'}(t)$. Then T is linear but unbounded. Indeed, let $x_n(t) = t^n$. Then $\|x_n\| = 1$ and $\|Tx_n\| = n$, i.e., T is not bounded.

Example 2.27 (Integration is Bounded)**.** Let $T : C[0,1] \rightarrow C[0,1], k \in C[0,1]^2$, $y = Tx$ via

$$y(t) = \int_0^1 k(t,\tau)x(\tau)d\tau.$$

Since k is continuous, $\exists \ k_0 > 0$ s.t.

$$|k(t,\tau)| \leq k_0.$$

So then

$$\begin{aligned} \|y\| &= \|Tx\| \\ &\leq \max_{t\in[0,1]} \int_0^1 |k(t,\tau)| \, \|x(t)\| \, d\tau \\ &\leq k_0 \, \|x\| \end{aligned}$$

Theorem 2.28 (Finite Dimension)**.** *Let X be normed and* $\dim X = n < \infty$*. Then every linear operator on X is bounded.*

Proof. Let $e_1, \ldots, e_n$ be a basis for X and let $x \in X$ be arbitrary and represented as

$$x = \sum_{i=1}^n \alpha_i e_i.$$

Then

$$\|Tx\| = \left\| \sum \right\|_{i=1}^n \alpha_i Te_i \leq k \sum_{i=1}^n |\alpha_i|,$$

where $k = \max_{i=1,\ldots,n} \|Te_i\|$. Also,

$$\sum_{i=1}^n |\alpha_i| \leq \frac{1}{c} \, \|x\| \, .$$

It follows that T is bounded.

□

Next, we illustrate a general method for computing bounds on operators.

Example 2.29 (Calculating Bounds on Operators)**.** Let

$$x = \sum_{i=1}^{n} \xi_i e_i.$$

Then

$$\begin{aligned} \|Tx\| &= \left\| \sum_{i=1}^{n} \xi_i T e_i \right\| \\ &\leq \sum_{i=1}^{n} |\xi_i| \, \|Te_i\| \\ &\leq \max_{k=1,\ldots,n} \|Te_k\| \sum_{i=1}^{n} |\xi_i| \, . \end{aligned}$$

Then using Theorem 2.9 we can conclude

$$\frac{1}{c} \|x\| = \frac{1}{c} \left\| \sum_{i=1}^{n} \xi_i e_i \right\| \geq \sum_{i=1}^{n} |\xi_i|$$

which implies

$$\|Tx\| \leq \gamma \|x\| \text{ where } \gamma = \frac{1}{c} \max_{k=1,\ldots,n} \|Te_k\| \, .$$

Definition (Continuity for Operators)**.** Let $T : \mathcal{D}(T) \overset{\text{into}}{\longrightarrow} Y$. Then T is said to be continuous at $x_0 \in \mathcal{D}(T)$ if

$$\forall \, \varepsilon > 0 \; \exists \, \delta > 0 \text{ s.t. } \|x - x_0\| < \delta \Rightarrow \|Tx - Tx_0\| < \varepsilon.$$

T is said to be continuous if it is continuous at every $x_0 \in \mathcal{D}(T)$.

Theorem 2.30 (Continuity and Boundedness)**.** *Let X, Y be vector spaces $\mathcal{D}(T) \subset X$, and $T : \mathcal{D}(T) \overset{into}{\longrightarrow} Y$ be linear.*

1. *T is continuous iff T is bounded*
2. *T is continuous at a single point $\Rightarrow$ T is continuous everywhere*

Proof.

1. The case $T = 0$ is trivial. Assume $T \neq 0$, then $\|T\| \neq 0$. Suppose T is bounded and let $x_0 \in \mathcal{D}(T), \varepsilon > 0$ be arbitrary. Then because T is linear, $\forall \, x \in \mathcal{D}(T)$ s.t.

$$\|x - x_0\| < \delta = \frac{\varepsilon}{\|T\|}$$

we have that

$$\|Tx - Tx_0\| = \|T(x - x_0)\| \leq \|T\| \, \|x - x_0\| < \|T\| \, \delta = \varepsilon.$$

Therefore T is continuous.

Conversely, suppose T is continuous at $x_0 \in \mathcal{D}(T)$. Then $\forall \, \varepsilon > 0 \; \exists \, \delta > 0$ s.t.

$$\|x - x_0\| \leq \delta \Rightarrow \|Tx - Tx_0\| \leq \varepsilon.$$

So choose any $y \in \mathcal{D}(T), y \neq 0$ and set

$$x = x_0 + \delta \frac{y}{\|y\|}.$$

Then

$$x - x_0 = \delta \frac{y}{\|y\|} \Rightarrow \|x - x_0\| = \delta,$$

so

$$\begin{aligned} \|Tx - Tx_0\| &= \|T(x - x_0)\| \\ &= \left\| T \left(\frac{\delta}{\|y\|} y \right) \right\| \\ &= \frac{\delta}{\|y\|} \|Ty\| \\ &\leq \varepsilon. \end{aligned}$$

Therefore we have

$$\|Ty\| \leq c \, \|y\| \text{ where } c = \frac{\varepsilon}{\delta}.$$

2. Above we only used continuity at a single point to show boundedness. Boundedness implies continuity.

□

Corollary 2.31. *Let* $T : \mathcal{D}(T) \overset{into}{\longrightarrow} Y$ *be a bounded, linear operator. Then*

1. $(x_n) \subset \mathcal{D}(T), x \in \mathcal{D}(T)$ *and* $x_n \to x \Rightarrow Tx_n \to Tx$

2. $\mathcal{N}(T)$ *is closed.*

Proof. 1. $\|Tx_n - Tx\| = \|T(x_n - x)\| \leq \|T\| \, \|x_n - x\| \to 0$

2. $\forall \, x \in \mathcal{N}(T) \; \exists \, (x_n) \subset \mathcal{N}(T)$ s.t.

$$x_n \to x \Rightarrow Tx_n \to T_x.$$

But $x_n \in \mathcal{N}(T) \Rightarrow Tx_n = 0, \; \forall \, n \Rightarrow Tx = 0 \Rightarrow x \in \mathcal{N}(T)$. Therefore $\mathcal{N}(T)$ is closed.

□

Compare to the notion of subadditivity (triangle inequality).

Remark 2.32 (Operator Norm is Submultiplicative)**.**

1. $\|TS\| \leq \|T\| \, \|S\|$
2. $\|T^n\| \leq \|T\|^n$

Remark 2.33 (Equality of Operators)**.** *Two operators T, S are said to be equal and we write $T = S$ if*

$$\mathcal{D}(T) = \mathcal{D}(S) \text{ and } Tx = Sx, \; \forall \; x \in \mathcal{D}(T).$$

Definition (Rescriction)**.** Let $T : \mathcal{D}(T) \xrightarrow{\text{into}} Y$ and let $B \subset \mathcal{D}(T)$. We define the restriction of T to B, denoted $T|_B : B \xrightarrow{\text{into}} Y$, as

$$T|_B x = Tx, \; \forall \; x \in B.$$

Definition (Extension)**.** Let $T : \mathcal{D}(T) \xrightarrow{\text{into}} Y$ and $\mathcal{D}(T) \subset M$. An extension of T is $\hat{T} : M \xrightarrow{\text{into}} Y$ s.t.

$$\hat{T}|_{\mathcal{D}}(T) = T$$

$$\hat{T}x = Tx, \; \forall \; x \in \mathcal{D}(T).$$

Theorem 2.34 (Bounded Linear Extension)**.** *Let $\mathcal{D}(T) \subset X$, X a normed space, Y a Banach space, and $T : \mathcal{D}(T) \xrightarrow{into} Y$ be linear. Then there is a unique bounded, linear extension $\hat{T}$ of T s.t.*

$$\left\| \hat{T} \right\| = \|T\|.$$

Proof. Let $x \in \overline{\mathcal{D}(T)}$. Then $\exists \; (x_n) \in \mathcal{D}(T)$ s.t. $x_n \to x$. Since T is linear and bounded we have

$$\|Tx_n - Tx\| = \|T(x_n - x)\| \leq \|T\| \, \|x_n - x\|,$$

so (Tx_n) is Cauchy. Since Y is Banach, (Tx_n) converges, say

$$Tx_n \to y \in Y.$$

In this way, we define

$$\hat{T}x = y.$$

Because limits are unique, $\hat{T}$ is uniquely defined $\forall \; x \in \overline{\mathcal{D}(T)}$. Of course $\hat{T}$ inherits linearity and

$$\hat{T}x = Tx, \; \forall \; x \in \mathcal{D}(T)$$

so $\hat{T}$ is a genuine extension.

Finally, we need to show $\hat{T}$ is bounded. Note

$$\|Tx_n\| \leq \|T\| \|x_n\|$$

and recall

$$x \mapsto \|x\|$$

is continuous. Therefore $Tx_n \to y = \hat{T}x$ and

$$\left\|\hat{T}x\right\| \leq \|x\| \|x\| .$$

Therefore $\hat{T}$ is bounded and $\left\|\hat{T}\right\| \leq \|T\|$. By definition, $\left\|\hat{T}\right\| \geq \|T\|$ so

$$\left\|\hat{T}\right\| = \|T\| .$$

□

2.5 Linear Functionals

Definition (Linear Functional)**.** A linear functional f is a linear operator with $\mathcal{D}(f) \subset X$ where X is vector space and $\mathcal{R}(f) \subset \mathbb{K}$ where $\mathbb{K}$ is the scalar field corresponding to X ($\mathbb{R}$ or $\mathbb{C}$). Then

$$f : \mathcal{D}(f) \overset{\text{into}}{\longrightarrow} \mathbb{K}.$$

Definition (Bounded Linear Functional)**.** Let $f : \mathcal{D}(f) \overset{\text{into}}{\longrightarrow} \mathbb{K}$ be a linear functional. Then f is said to be bounded if $\exists\ c > 0$ s.t.

$$|f(x)| \leq c \|x\| ,\ \forall\ x \in \mathcal{D}(f) .$$

Definition (Norm of a Linear Functional)**.** Let f be a linear functional. Then

$$\|f\| = \sup_{\substack{x \in \mathcal{D}(f) \\ x \neq 0}} \frac{|f(x)|}{\|x\|} = \sup_{\substack{x \in \mathcal{D}(f) \\ \|x\| = 1}} |f(x)| ,$$

so

$$|f(x)| \leq \|f\| \|x\| .$$

Theorem 2.35 (Continuity and Boundedness)**.** *Let f be a linear functional. f is continuous iff f is bounded.*

Example 2.36 (Dot Product)**.** Let $a \in \mathbb{R}, a \neq 0$ be fixed and let $f : \mathbb{R}^n \to \mathbb{R}$ be defined as

$$f(x) = x \cdot a.$$

Then f is linear functional and

$$|f(x)| \leq \|x\| \, \|a\| \, .$$

On the other hand, if we take $x = a$

$$|f(x)| = \|a\|^2 \Rightarrow \|f\| \geq \|a\| \, .$$

Example 2.37 (Integral)**.** Define $f : C[a, b] \to \mathbb{R}$ by

$$f(x) = \int_a^b x(t)dt.$$

Then

$$|f(x)| \leq (b - a) \|x\| \, .$$

Further, if $x = 1$

$$|f(x)| = b - a \Rightarrow \|f\| = b - a.$$

Example 2.38 (Point Evaluation)**.** Let $t_0 \in [a, b]$ be fixed and define $f : C[a, b] \to \mathbb{R}$ by

$$f(x) = x(t_0).$$

Choosing the sup-norm, we see

$$|f(x)| = |x(t_0)| \leq \|x\| \, ,$$

and since $f(1) = 1 = \|1\|$, we have

$$\|f\| = 1.$$

Example 2.39 (Point Evaluation in $\mathcal{L}^2$)**.** Let $f : \mathcal{L}^2[a, b] \to \mathbb{R}$ by

$$f(x) = x(a).$$

Choose the $\mathcal{L}^2$ norm, and let $x \in \mathcal{L}^2[a, b]$ be such that $x(a) = 1$ but then x rapidly decreases to 0. Then $f(x) = 1$ but

$$\|x\| = \left(\int_a^b |x(t)|^2 \, dt \right)^{\frac{1}{2}}$$

and for any $\varepsilon > 0$ we can choose an x such that $\|x\| < \varepsilon$. Therefore there is no such $c > 0$ s.t.

$$|f(x)| = 1 \leq c \, \|x\| \, .$$

So point evaluation in this space with this norm is unbounded.

Example 2.40 (Inner Product in ℓ^2)**.** Fix $a \in \ell^2$ and define $f : \ell^2 \to \mathbb{R}$ by

$$f(x) = \sum_{i=1}^{\infty} x_i a_i = \langle x, a \rangle .$$

Then

$$\begin{aligned} |f(x)| &\leq \sum_{i=1}^{\infty} |x_i a_i| \\ &\leq \sqrt{\sum_{i=1}^{\infty} x_i^2} \sqrt{\sum_{i=1}^{\infty} a_i^2} \\ &= \|x\| \, \|a\| . \end{aligned}$$

Definition (Algebraic Dual)**.** Let X be a vector space. Then we denote X^* as the space of linear functionals on X. X^{**} represents the dual of X^*, et cetera.

Space	General Element	Value at a Point
X	x	n/a
X^*	f	$f(x)$
X^{**}	g	$g(f)$

Definition (Isomorphism = Bijective Isometry)**.** Let X, Y be vector spaces over the same scalar field $\mathbb{K}$. Then an isomorphism $T : X \to Y$ is a bijective mapping which preserves the two algebraic operations of vector spaces, i.e.

$$T(\alpha x + \beta y) = \alpha T x + \beta T y.$$

So T is a bijective linear operator and we say X, Y are isomorphic.

If in addition X, Y are normed spaces, then T must also preserve the norms

$$\|x\| = \|Tx\| .$$

We can obtain $g \in X^{**}$ picking a fixed $x \in X$ and setting

$$g(f) = g_x(f) = f(x), \ \forall \ f \in X^*.$$

This g is linear, and $x \in X \Rightarrow g_x \in X^{**}$.

Definition (Canonical Mapping/Embedding)**.** Define $C : X \to X^{**}$ by

$$x \mapsto g_x.$$

Then C is injective and linear. Therefore C is an isomorphism of X and $\mathcal{R}(C) \subset X^{**}$.

An interesting and important problem is when $\mathcal{R}(C) = X^{**}$, i.e., X, X^{**} are isomorphic.

Definition (Embeddable)**.** Let X, Y be vector spaces. X is said to be embeddable in Y if it is isomorphic with a subspace of Y.

From Definition 2.5 we can see X is always embeddable in X^{**}. C is also called the canonical embedding of X into X^{**}.

Definition (Algebraic Reflexive)**.** If the canonical mapping C is surjective (and hence bijective) then

$$\mathcal{R}(C) = X^{**}$$

and X is said to be algebraically reflexive.

Problem 13. If Y is a subspace of a vector space X and f is a linear functional on X such that $f(Y)$ is not the whole scalar field $\mathbb{K}$, show

$$f(y) = 0, \;\forall\, y \in Y.$$

Solution. Suppose that $f(y) = b \neq 0$ for some $y \in Y$. For arbitrary $a \in R$ we have $\frac{a}{b}y \in Y$ and $f(\frac{a}{b}y) = a$. It follows that $f(Y) = R$; a contradiction. □

Problem 14. Let $f \neq 0$ be a linear functional on a vector space X, and let $x_0 \in X - \mathcal{N}(f)$ be fixed. Show that every $x \in X$ has a unique representation

$$x = \alpha x_0 + y, \text{ some } y \in \mathcal{N}(f).$$

Solution. Let $f \neq 0$ and consider $x_0 \in X - N(f)$. Then $f(x_0) = c \neq 0$. Let $a = f(x) = f(\frac{a}{c}x_0)$. It follows that $x - \frac{a}{c}x_0 \in N(f)$. We can write

$$x = \frac{a}{c}x_0 + y \text{ , where } y = x - \frac{a}{c}x_0 \in N(f).$$

□

2.6 Finite Dimensional Case

Let X, Y be finite dimensional vector spaces over the same field $\mathbb{K}$, and let $T : X \to Y$ be a linear operator. Let $E = \{e_1, \ldots, e_n\}, B = \{b_1, \ldots b_n\}$ be bases for X, Y respectively. Then $x \in X$ and $y = Tx \in Y$ and $Te_k \in Y, \;\forall\, k$ have the unique representations

$$\begin{aligned} x &= \sum_{k=1}^{n} \xi_k e_k \\ Te_k &= \sum_{i=1}^{n} \tau_{ik} b_i \\ y &= \sum_{i=1}^{n} \eta_i b_i. \end{aligned}$$

Then calculate

$$\begin{aligned} y &= Tx \\ &= T\left(\sum_{k=1}^{n} \xi_k e_k\right) \\ &= \sum_{k=1}^{n} \xi_k T e_k \end{aligned}$$

which implies

$$\begin{aligned} y = \sum_{i=1}^{n} \eta_i b_i &= \sum_{k=1}^{n} \xi_k \left(\sum_{i=1}^{n} \tau_{ik} b_i\right) \\ &= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} \tau_{ik} \xi_k\right) b_i \end{aligned}$$

yielding

$$\eta_i = \sum_{k=1}^{n} \tau_{ik} \xi_k.$$

So if we label

$$T_{EB} = (\tau_{ik}), \bar{x} = (\xi_k), \bar{y} = (\eta_i)$$

then

$$\bar{y} = T_{EB} \bar{x}.$$

We say T_{EB} represents T with respect to the bases E, B.

Remark 2.41 (Finite Dimensional Linear Operators are Matrices)**.** *The above argument shows that every f.d.l.o. can be represented as a matrix.*

2.6.1 Linear Functionals

Let X be a vector space, $n = \dim X$, and let $\{e_1, \ldots, e_n\}$ be a basis for X. Label

$$\alpha_i = f(e_i).$$

Then

$$f(x) = f\left(\sum_{i=1}^{n} \xi_i e_i\right) = \sum_{i=1}^{n} \xi_i \alpha_i.$$

(f is uniquely determined by its values α_i at the n basis vectors of X).

Definition (Dual Basis)**.** Define f_k by

$$f_k(e_i) = \delta_{ik},\ \forall\ k = 1, \dots, n.$$

Then

$$\{f_1, \dots, f_n\} \overset{\text{bij}}{\leftrightarrow} \{e_1, \dots, e_n\}\,.$$

Theorem 2.42 (Dimension of X^*)**.** *Let X be a vector space, $n = \dim X$, and let $E = \{e_1, \dots, e_n\}$ be a basis for X. Then $\{f_1, \dots, f_k\}$ is a basis for X^* and $\dim X^* = n$. So then if $f \in X^*$ then we can represent*

$$f = \sum_{i=1}^{n} \alpha_i f_i.$$

Proof. See Theorem 2.41. □

Lemma 2.43 (Zero Vector)**.** *Let X be a fdvs. If $x_0 \in X$ has the property that $f(x_0) = 0,\ \forall\ f \in X^*$ then $x_0 = 0$.*

Proof. Let $\{e_1, \dots, e_n\}$ be a basis for X and let

$$x_0 = \sum_{i=1}^{n} \xi_{0i} e_i.$$

Then

$$f(x_0) = \sum_{i=1}^{n} \xi_{0i} \alpha_i.$$

By assumption $f(x_0) = 0,\ \forall\ f \in X^* \Rightarrow \xi_{0i} = 0,\ \forall\ i = 1, \dots, n.$ □

Theorem 2.44 (Algebraic Reflexivity)**.** *A fdvs X is algebraically reflexive, i.e., X is isomorphic to X^{**}.*

Proof. By construction $C : X \overset{\text{into}}{\longrightarrow} X^{**}$ is linear. Then

$$Cx_0 = 0 \Rightarrow (Cx_0)(f) = g_{x_0}(f) = f(x_0) = 0,\ \forall\ f \in X^*$$

and therefor $x_0 = 0$ by the previous lemma. Therefore C has an inverse $C^{-1} : \mathcal{R}(C) \to X$. Therefore $\dim \mathcal{R}(C) = \dim X$. Therefor C is an isomorphism and X is algebraically reflexive. □

Problem 15. Find a basis for the null space of functional $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x) = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \alpha_3 \xi_3$$

where $\alpha_1 \neq 0$.

Solution. Let $(\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1 \neq 0$ be a fixed point in R^3 and let $f(x) = \sum a_i x_i$. The vectors $(-\frac{\alpha_2}{\alpha_1}, 1, 0)$ and $(-\frac{\alpha_3}{\alpha_2}, 0, 1)$ form a basis for $N(f)$. □

2.7 Dual Space

Definition $(B(X,Y))$**.** Let X, Y be vector spaces. Then we define $B(X,Y)$ as the set of all bounded linear operators from X to Y.

Remark 2.45 ($B(X,Y)$ is a Vector Space)**.** $(\alpha S + \beta T)x = \alpha Sx + \beta Tx$

Theorem 2.46 ($B(X,Y)$ is a Normed Space)**.** *Let X, Y be normed spaces. The vector space $B(X,Y)$ is also a normed space with norm*

$$\|T\| = \sup_{\substack{x\in X\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x\in X\\ \|x\|=1}} \|Tx\|.$$

Theorem 2.47 ($B(X,Y)$ is a Banach Space)**.** *If in the assumptions of the previous proof Y is Banach, then $B(X,Y)$ is also a Banach space.*

Proof. Let (T_n) be an arbitrary Cauchy sequence in $B(X,Y)$. Then

$$\forall\, \varepsilon > 0\ \exists\ N \text{ s.t. } \|T_n - T_m\| < \varepsilon,\ \forall\ m, n > N.$$

Therefore $\forall\ x \in X,$ and $m, n > N$

$$\begin{aligned} \|T_n x - T_m x\| &= \|(T_n - T_m)x\| \\ &\leq \|T_n - T_m\| \|x\| \\ &< \varepsilon \|x\|. \end{aligned}$$

So fix $x \in X$. We see $\forall\ x \in X$ that $(T_n x) \subset Y$ is Cauchy. Y is complete so $\exists\ y \in Y$ s.t. $T_n x \to y$. We need to construct a limit for (T_n). Define $T : X \to Y$ by

$$Tx = y.$$

By construction, T is linear

$$\begin{aligned} \lim_{n\to\infty} T_n(\alpha x + \beta y) &= \lim_{n\to\infty} (\alpha T_n x + \beta T_n x) \\ &= \alpha \lim_{n\to\infty} T_n x + \beta \lim_{n\to\infty} T_n x \\ &= \alpha Tx + \beta Tx. \end{aligned}$$

The following

$$\begin{aligned} \|T_n x - Tx\| &= \left\| T_n x - \lim_{m\to\infty} T_m x \right\| \\ &= \lim_{m\to\infty} \|T_n x - T_m x\| \\ &\leq \varepsilon \|x\| \end{aligned}$$

implies that $T_n - T$ is bounded. Since $T_n \in B(X, Y) \Rightarrow T_n$ is bounded and

$$T = T_n - (T_n - T)$$

T is also bounded. Therefore $T \in B(X, Y)$.

Finally,

$$\frac{\|T_n x - Tx\|}{\|x\|} \leq \varepsilon$$

implies

$$\|T_n - T\| \leq \varepsilon \Rightarrow \|T_n - T\| \to 0.$$

□

Definition (Normed Dual Space X')**.** Let X be a n.s.. Then the set of all bounded linear functionals on X constitutes a n.s. with norm

$$\|f\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{\|x\|} = \sup_{\substack{x \in X \\ \|x\|=1}} |f(x)|.$$

This is the normed dual of X.

Theorem 2.48 (X' is a Banach Space)**.** *Let X be a n.s. and let X' be the dual of X. Then X' is a Banach space.*

Proof. See Theorem 2.47. □

Example 2.49 ($\mathbb{R}^n$)**.** The dual of $\mathbb{R}^n$ is $\mathbb{R}^n$.

Proof. We claim $\mathbb{R}^{n*} = \mathbb{R}^{n\prime}$ and $\forall\ f \in \mathbb{R}^{n*}$ we have

$$f(x) = \sum_{k=1}^{n} \xi_k \alpha_k, \alpha_k = f(e_k).$$

So then

$$|f(x)| \leq \sum_{k=1}^{n} |\xi_k \alpha_k| \leq \|x\| \sqrt{\sum_{k=1}^{n} \alpha_k^2},$$

which implies

$$\|f\| \leq \sqrt{\sum_{k=1}^{n} \alpha_k^2}.$$

If we choose $x = (\alpha_k)$ then

$$|f(x)| = \sqrt{\sum_{k=1}^{n} \alpha_k^2} \Rightarrow \|f\| = \sqrt{\sum_{k=1}^{n} \alpha_k^2}.$$

So take the onto mapping from $\mathbb{R}^{n\prime}$ to $\mathbb{R}^n$

$$f \mapsto \alpha = (\alpha_k) = (f(e_k))$$

which is norm-preserving, linear, and a bijection (an isomorphism), which completes the proof. □

Problem 16.

1. Show $\ell^{1\prime} = \ell^\infty$
2. Show $\ell^{p\prime} = \ell^q$ where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$

Solution.

1. A Schauder basis for ℓ^1 is (e_k), where $e_k = (\delta_{ki})$. Then every $x \in \ell^1$ has a unique representation

$$x = \sum_{k=1}^{\infty} \xi_k e_k.$$

We consider any $f \in \ell^{1\prime}$, where $\ell^{1\prime}$ is the dual space of ℓ^1. Since f is linear and bounded,

$$f(x) = \sum_{k=1}^{\infty} \xi_k \gamma_k \text{ , where } \gamma_k = f(e_k).$$

It is easy to see that $(\gamma_k) \in \ell^\infty$.
On the other hand, for every $b = (\beta_k) \in \ell^\infty$,

$$g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k \text{ , where } x = (\xi_k) \in \ell^1$$

is a bounded linear functional on ℓ^1, i.e., $g \in \ell^{1\prime}$.
It is also easy see that $\|f\| = \|c\|_\infty$, where $c = (\gamma_k) \in \ell^\infty$. It follows that the bijective linear mapping of $\ell^{1\prime}$ onto ℓ^∞ defined by $f \mapsto c$ is an isomorphism.

2. Take the same Schauder basis and representation for x and f as in 1. Let q be the conjugate of p and consider $x_n = (\xi_k^{(n)})$ with

$$\xi_k^{(n)} = \frac{|\gamma_k|^q}{\gamma_k} \text{ , if } k \leq n \text{ and } \gamma_k \neq 0,$$

and

$$\xi_k^{(n)} = 0 \text{ if } k > n \text{ or } \gamma_k = 0.$$

Then we have

$$f(x_n) = \sum_{k=1}^{\infty} \xi_k^{(n)} \gamma_k = \sum_{k=1}^{n} |\gamma_k|^q \leq \|f\| (\sum_{k=1}^{n} |\gamma_k|^q)^{\frac{1}{p}}.$$

Using $1 - \frac{1}{p} = \frac{1}{q}$ we get

$$(\sum_{k=1}^{n} |\gamma_k|^q)^{\frac{1}{q}} \leq \|f\|.$$

Since n is arbitrary, letting $n \to \infty$, we obtain

$$(\sum_{k=1}^{\infty} |\gamma_k|^q)^{\frac{1}{q}} \leq \|f\|,$$

and therefore $(\gamma_k) \in \ell^q$.
Conversely, for any $b = (\beta_k) \in \ell^q$ we may define g on ℓ^p by

$$g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k \text{ , where } x = (\xi_k) \in \ell^p.$$

Then g is linear, and boundedness follows from the Holder inequality. Hence $g \in \ell^{p'}$.
From the Holder inequality we get

$$|f(x)| \leq \|x\| (\sum_{k=1}^{\infty} |\gamma_k|^q)^{\frac{1}{q}}.$$

It follows that $\|f\| = \|c\|_q$, where $c = (\gamma_k) \in \ell^q$ and $\gamma_k = f(e_k)$. The mapping of $\ell^{p'}$ onto ℓ^q defined by $f \mapsto c$ is linear, bijective, and norm-preserving, so it is an isomorphism.

□

3 Hilbert Spaces

Definition (Inner Product)**.** Let X be vector space with scalar field $\mathbb{K}$. $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$ is called an inner product if

1. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
2. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
3. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
4. $\langle x, x \rangle \geq 0, \; \langle x, x \rangle = 0$ iff $x = 0$

Definition (Hilbert Space)**.** A Hilbert space is a complete inner product space.

Remark 3.1.

1. *A vector space X with inner product $\langle \cdot, \cdot \rangle$ has the induced norm and metric:*

$$\|x\| = \langle x, x \rangle$$
$$d(x, y) = \|x - y\| .$$

2. *If X is real then $\langle x, y \rangle = \langle y, x \rangle$.*
3. *From the definition of inner product, we obtain*

$$\begin{aligned} \langle \alpha x + \beta y, z \rangle &= \alpha \langle x, z \rangle + \beta \langle y, z \rangle \\ \langle x, \alpha y \rangle &= \overline{\alpha} \langle x, y \rangle \\ \langle x, \alpha y + \beta z \rangle &= \overline{\alpha} \langle x, y \rangle + \overline{\beta} \langle x, z \rangle \end{aligned}$$

   So the inner product is linear in the first argument and conjugate linear in the second argument.

4. *Parallelogram Equality.*

$$\|x + y\|^2 + \|x - y\|^2 = 2 \left(\|x\|^2 + \|y\|^2 \right) \qquad (5)$$

   Any norm which is induced from an inner product must satisfy this equality.

5. *Orthogonality. $x, y \in X$ are said to be* **orthogonal** *if $\langle x, y \rangle = 0$ and we write*

$$x \perp y.$$

   For subsets $A, B \subset X$ we say $A \perp B$ if

$$a \perp b \;\forall\; a \in A, b \in B.$$

Example 3.2.

1. $\mathbb{R}^n$ with inner product $\langle x, y\rangle = x^T y$.
2. $\mathbb{C}^n$ with inner product $\langle x, y\rangle = x^T \overline{y}$.
3. Real $\mathcal{L}^2[a,b]$ with
$$\langle x, y\rangle = \int_a^b x(t)y(t)dt.$$
4. Complex $\mathcal{L}^2[a,b]$ with
$$\langle x, y\rangle = \int_a^b x(t)\overline{y(t)}dt.$$
5. ℓ^2 with inner product
$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}.$$

Problem 17.

1. Show ℓ^p with $p \neq 2$ is not a Hilbert space.
2. Show $C[a,b]$ is not a Hilbert space.

Solution.

1. Consider $x = (1, 1, 0, ..., 0, ..)$ and $y = (1, -1, 0, ..., 0, ...)$. Then $x, y \in l^p$ and $\|x\|_p = \|y\|_p = 2^{\frac{1}{p}}$, and $\|x+y\|_p = \|x-y\|_p = 2$. It follows that if $p \neq 2$, then the paralelogram equality is not satisfied.
2. Let $x(t) = \frac{t-a}{b-a}$ and $y(t) = 1 - \frac{t-a}{b-a}$. Then we have $\|x\| = \|y\| = \|x+y\| = \|x-y\| = 1$ and the paralelogram equality is not satisfied.

□

Problem 18. Show that $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm.

Solution. It follows from the definition of the inner product that $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$. Also, we have using the definiton of the inner product that

$$\|\alpha x\| = (\langle \alpha x, \alpha x\rangle)^{\frac{1}{2}} = (\alpha\bar{\alpha} \langle x, x\rangle)^{\frac{1}{2}} = (|\alpha|^2 \langle x, x\rangle)^{\frac{1}{2}} = |\alpha|\|x\|.$$

Concerning the triangle inequality we get

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \|x\|^2 + 2Re\,\langle x, y\rangle + \|y\|^2$$
$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2,$$

from which the result follows. □

Lemma 3.3 (Continuity of Inner Product)**.** *If in an inner product space $y_n \to y$ and $x_n \to x$ then*

$$\langle x_n, y_n \rangle \to \langle x, y \rangle .$$

Theorem 3.4 (Completion)**.** *For any inner product space X there is a Hilbert space H and isomorphism A from X onto a dense subspace $W \subset H$. The space H is unique up to isomorphisms.*

Theorem 3.5 (Subspace)**.** *Let $Y \subset H$ with H a Hilbert space. Then*

1. *Y is complete iff Y is closed in H.*
2. *If Y is finite dimensional then Y is complete.*
3. *If H is separable then Y is separable.*

Definition (Convex Subset)**.** $M \subset X$ is said to be convex if

$$\alpha x + (1-\alpha) y \in M, \ \forall \ \alpha \in [0,1], a, b \in M.$$

Theorem 3.6. *Let X be an inner product space and $M \neq \emptyset$ a convex subset which is complete. Then $\forall \ x \in X \ \exists! \ y \in M$ s.t.*

$$\delta = \inf_{\hat{y} \in M} \|x - \hat{y}\| = \|x - y\| .$$

Proof.

1. Existence. By definition of inf,

$$\exists \ (y_n) \text{ s.t. } \delta_n \to \delta, \text{ where } \delta_n = \|x - y_n\| .$$

We will show (y_n) is Cauchy. Write $v_n = y_n - x$. Then $\|v_n\| = \delta_n$ and

$$\begin{aligned} \|v_n + v_m\| &= \|y_n + y_m - 2x\| \\ &= 2 \left\| \frac{1}{2}(y_n + y_m) - x \right\| \geq 2\delta \end{aligned}$$

Furthermore, $y_n - y_m = v_n - v_m, \Rightarrow$

$$\begin{aligned} \|y_n - y_m\|^2 &= \|v_n - v_m\|^2 \\ &= -\|v_n + v_m\|^2 + 2(\|v_n\|^2 + \|v_m\|) \\ &\leq -(2\delta)^2 + 2(\delta_n^2 + \delta_m^2). \end{aligned}$$

Since M is complete, $y_n \to y \in M$. Then $\|x - y\| \geq \delta$. Also, we have

$$\begin{aligned} \|x - y\| &\leq \|x - y_n\| + \|y_n - y\| \\ &= \delta_n + \|y_n - y\| \\ &\to \delta + 0 \end{aligned}$$

therefore $\|x - y\| = \delta$.

2. Uniqueness. Suppose that two such elements exist, $y_1, y_2 \in M$ and from above,

$$\|x - y_1\| = \delta = \|x - y_2\| .$$

By (5) we have

$$\begin{aligned}
\|y_1 - y_2\|^2 &= \|(y_1 - x) + (x - y_2)\|^2 \\
&= 2\|y_1 - x\|^2 + 2\|y_2 - x\|^2 - \|(y_1 - x) + (y_2 - x)\|^2 \\
&= 2\delta^2 + 2\delta^2 - 2^2 \left\| \frac{1}{2}(y_1 + y_2) - x \right\|^2 \\
&\leq 0
\end{aligned}$$

since

$$2^2 \left\| \frac{1}{2}(y_1 + y_2) - x \right\|^2 \geq 4\delta^2.$$

Therefore $y_1 = y_2$.

□

Lemma 3.7 (Orthogonality)**.** *With the setup of the previous theorem, $z = x - y$ is orthogonal to M.*

Proof. Suppose $z \perp M$ were false, then $\exists\ w \in M$ s.t.

$$\langle z, w \rangle = \beta \neq 0.$$

and so $w \neq 0$. For any α,

$$\begin{aligned}
\|z - \alpha w\|^2 &= \langle z - \alpha w, z - \alpha w \rangle \\
&= \langle z, z \rangle - \overline{\alpha} \langle z, w \rangle - \alpha \left[\langle w, z \rangle - \overline{\alpha} \langle w, w \rangle \right] \\
&= \|z\|^2 - \overline{\alpha}\beta - \alpha \left[\beta - \overline{\alpha} \langle w, w \rangle \right].
\end{aligned}$$

If we choose

$$\overline{\alpha} = \frac{\overline{\beta}}{\langle w, w \rangle},$$

and continue the equality above,

$$\begin{aligned}
\|z - \alpha w\|^2 &= \delta^2 - \frac{|\beta|^2}{\langle w, w \rangle} - \alpha [0] \\
&< \delta^2
\end{aligned}$$

contradicting the minimality of y, δ. Therefore we must have $z \perp M$. □

Definition (Direct Sum)**.** A vector space X is said to be **direct sum** of subspaces Y and Z, written

$$X = Y \oplus Z,$$

if $\forall\ x \in\ \exists!\ y \in Y, z \in Z$ s.t.

$$x = y + z.$$

Definition (Orthogonal Complement)**.** Let Y be a closed subspace of X. Then the orthogonal complement of Y in X is

$$Y^{\perp} = \{z \in X \text{ s.t. } z \perp Y\}.$$

Theorem 3.8 (Direct Sum)**.** *Let Y be a closed subspace of H. Then*

$$H = Y \oplus Y^{\perp}.$$

Proof.

1. Existence. H is complete and Y is closed implies Y is complete. Further, we know Y is convex. Therefore $\forall\ x \in H\ \exists\ y \in Y$ s.t.

$$x = y + z, \text{ where } z \in Y^{\perp}.$$

2. Uniqueness. Assume $x = y + z = y_1 + z_1$. Then

$$y - y_1 \in Y$$

$$z - z_1 \in Y^{\perp}.$$

But $Y \cap Y^{\perp} = \{0\} \Rightarrow y - y_1 = z - z_1 = 0$.

□

Definition (Orthogonal Projection)**.** Let Y be a closed subspace of H, so $H = Y \oplus Y^{\perp}$, and let

$$x = y + z$$

$$y \in Y$$

$$z \in Y^{\perp}.$$

Then the orthogonal projection onto Y is $P : H \to Y$ given by

$$Px = y.$$

Clearly P is bounded. Further P is idempotent, that is, $P^2 = P$, which we call **idempotent**.

Definition (Annihilator)**.** Let X be an inner product space and let M be a nonempty subset of X. Then the annihilator of M is

$$M^{\perp} = \{x \in X \text{ s.t. } x \perp M\} = \{x \in X \text{ s.t. } \langle x, v\rangle = 0 \ \forall \ v \in M\}.$$

Theorem 3.9. *Let M, X be as the definition above and denote $(M^{\perp})^{\perp} = M^{\perp\perp}$. Then*

1. *$M^{\perp}$ is a vector space.*
2. *$M^{\perp}$ is closed.*
3. *$M \subset M^{\perp\perp}$.*

Theorem 3.10. *If M is a closed subspace of a Hilbert space H then*

$$M^{\perp\perp} = M.$$

Lemma 3.11 (Dense Set)**.** *Let $M \neq \emptyset$ be a subset of a Hilbert space H. Then*

$$\overline{\mathit{span}\,(M)} = H \text{ iff } M^{\perp} = \{0\}.$$

Proof.

1. $\Rightarrow$. Assume $x \in M^{\perp}, V = \text{span}\,(M)$ is dense in H. Then $x \in \overline{V} = H \Rightarrow \ \exists \ (x_n) \subset V$ s.t. $x_n \to x$. Now $x \in M^{\perp}$ and $M^{\perp} \perp V \Rightarrow \langle x_n, x\rangle = 0$. But $\langle \cdot, \cdot \rangle$ is continuous $\Rightarrow \langle x_n, x\rangle \to \langle x, x\rangle = 0 \Rightarrow x = 0 \Rightarrow M^{\perp} = \{0\}$, since $x \in M^{\perp}$ was arbitrary.

2. $\Leftarrow$. Suppose $M^{\perp} = \{0\}$ and write $V = \text{span}\,(M)$. Then

$$\begin{aligned} x \perp V &\Rightarrow x \perp M \\ &\Rightarrow x \in M^{\perp} \\ &\Rightarrow x = 0 \\ &\Rightarrow V^{\perp} = \{0\} \\ &\Rightarrow \overline{V} = H. \end{aligned}$$

□

Definition (Orthonormal Set/Sequence)**.** M is said to be an orthogonal set if $\forall \ x, y \in M$

$$x \neq y \Rightarrow \langle x, y\rangle = 0.$$

M is said to be orthonormal if $\forall \ x, y \in M$

$$\langle x, y\rangle = \delta_{xy} = \begin{cases} 0 & x \neq y \\ 1 & x = y \end{cases}.$$

If M is countable and orthogonal (resp. orthonormal) then we can write $M = (x_n)$ and we say (x_n) is an orthogonal (resp. orthonormal) sequence.

Remark 3.12 (Pythagorean Relation, Linear Independence)**.** *Let $M = \{x_1, \ldots, x_n\}$ be an orthogonal set. Then*

1.
$$\left\| \sum_{i=1}^{n} x_i \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2 .$$

2. *M is linearly independent.*

Example 3.13 (Orthonormal Sequences)**.**

1. $\{(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1,0),(0,\ldots,0,1)\} \subset \mathbb{R}$.
2. $(e_n) \subset \ell^2$ where $e_n = \delta_{ni}$.
3. Let $X = C[0,2\pi]$ with $\langle x,y\rangle = \int_0^{2\pi} x(t)y(t)dt$. Then the functions
$$(\sin nt), n = 1,2,\ldots,(\cos nt), n = 0,1,2,\ldots$$
form an orthogonal sequence.

Remark 3.14 (Unique Representation with Orthonormal Sequences)**.** *Let X be an inner product space and let (e_k) be an orthonormal sequence in X, and suppose $x \in span(\{e_1,\ldots,e_n\})$ where n is fixed. Then we can represent*

$$x = \sum_{k=1}^{n} \alpha_k e_k \quad \Rightarrow \quad \langle x, e_i\rangle = \left\langle \left[\sum_{k=1}^{n} \alpha_k e_k\right], e_i \right\rangle = \alpha_i$$
$$\Rightarrow \quad x = \sum_{k=1}^{n} \langle x, e_k\rangle e_k.$$

Now let $x \in X$ be arbitrary and take $y \in span(\{e_1,\ldots,e_n\})$, where

$$y = \sum_{k=1}^{n} \langle x, e_k\rangle e_k.$$

Define z by setting $x = y + z \Rightarrow z \perp y$ because

$$\begin{aligned}
\langle z, y\rangle &= \langle x - y, y\rangle \\
&= \langle x, y\rangle - \langle y, y\rangle \\
&= \left\langle x, \left[\sum \langle x, e_k\rangle e_k\right]\right\rangle - \|y\|^2 \\
&= \sum \langle x, \langle x, e_k\rangle e_k\rangle - \|y\|^2 \\
&= \sum \overline{\langle x, e_k\rangle} \langle x, e_k\rangle - \sum |\langle x, e_k\rangle|^2 \\
&= 0.
\end{aligned}$$

Then

$$\begin{aligned}
\Rightarrow \quad \|x\|^2 &= \|y\|^2 + \|z\|^2 \\
\Rightarrow \quad \|z\|^2 &= \|x\|^2 - \sum |\langle x, e_k\rangle|^2 \geq 0 \\
\Rightarrow \quad \textstyle\sum_{k=1}^{n} |\langle x, e_k\rangle|^2 &\leq \|x\|^2 \\
\Rightarrow \quad \textstyle\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 &\leq \|x\|^2
\end{aligned}$$

Remark 3.15 (Bessel's Inequaltiy)**.** *Let X be an inner product space and let (e_k) be an orthonormal sequence in X. Then the last inequality in the above remark is called* **Bessel's Inequality***:*

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq \|x\|^2 .$$

Definition (Fourier Coefficients)**.** The sequence $(\langle x, e_k\rangle)$ is called the Fourier coefficients of x w.r.t. (e_k).

Problem 19 (Fourier Coefficients are Minimizers). Let $\{e_n, \ldots, e_n\}$ be an orthonormal set in an inner product space X (n is fixed). Let $x \in X$ be an arbitrary, fixed element and let $y = \sum_{k=1}^{n} \beta_k e_k$. Then $\|x - y\|$ depends on $\beta_1, \ldots, \beta_n$. Show by direct calculation that $\|x - y\|$ is minimized iff $\beta_i = \langle x, e_i\rangle$, $\forall\ i = 1, \ldots, n$.

Solution. Let $\gamma_i = \langle x, e_i\rangle$, and $y = \sum \beta_i e_i$. Then

$$\|x - y\|^2 = \left\langle x - \sum \beta_i e_i, x - \sum \beta_i e_i \right\rangle = \|x\|^2 - \sum \bar{\beta}_i \gamma_i - \sum \beta_i \bar{\gamma}_i + \sum |\beta_i|^2$$

$$= \|x\|^2 - \sum |\gamma_i|^2 + \sum |\beta_i - \gamma_i|^2$$

and this is minumum for given x and e_is iff $\beta_i = \gamma_i$. □

Problem 20 (Gramm-Schmidt). Orthonormalize the first three terms of the sequence $(1, t, t^2, t^3, \ldots)$ on the interval $[-1, 1]$ where

$$\langle x, y\rangle = \int_{-1}^{1} x(t)y(t)dt.$$

Solution. Let $f_1(t) = 1$, then $e_1(t) = \frac{f_1(t)}{\|f_1(t)\|} = \frac{1}{\sqrt{2}}$. Let $f_2(t) = t$. We have that $\langle f_2(\cdot), e_1(\cdot)\rangle = 0$, so we just need to normalize $f_2(t)$ to get $e_2(t) = \sqrt{\frac{3}{2}}t$. Let $f_3(t) = t^2$. Easy calculation shows that $\langle f_3(\cdot), e_1(\cdot)\rangle = \frac{\sqrt{2}}{3}$ and $\langle f_3(\cdot), e_2(\cdot)\rangle = 0$. Then $f_3(t) = \langle f_3(\cdot), e_2(\cdot)\rangle\, e_2(t) = t^2 - \frac{1}{3}$. Normalizing this quantity we get $e_3(t) = \sqrt{\frac{5}{8}}(3t^2 - 1)$. □

Theorem 3.16 (Convergence)**.** *Let H be a Hilbert space and let $(e_n) \subset H$ be an orthonormal sequence. Consider*

$$\sum_{k=1}^{\infty} \alpha_k e_k. \tag{6}$$

1. *The series (6) converges in the induced norm of H iff*

$$\sum_{k=1}^{\infty} |\alpha_k|^2 < \infty.$$

2. *If (6) converges to x, then $\alpha_k = \langle x, e_k \rangle$ and*

$$x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k.$$

3. *For any $x \in H$ the series (6) with $\alpha_k = \langle x, e_k \rangle$ converges (in the norm of H).*

Proof. Let

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \alpha_k e_k \\ \sigma_n &= \sum_{k=1}^{n} |\alpha_k|^2 . \end{aligned}$$

1. For $n > m$

$$\begin{aligned} \|s_n - s_m\|^2 &= \left\| \sum_{k=m+1}^{n} \alpha_k e_k \right\|^2 \\ &= \sum_{k=m+1}^{n} |\alpha_k|^2 \\ &= \sigma_n - \sigma_m. \end{aligned}$$

   Hence (s_n) is Cauchy in H iff $\sigma_n)$ is Cauchy in $\mathbb{R}$.

2. Note $\langle s_n, e_i \rangle = \alpha_i,\ \forall\ i = 1, \ldots, k \leq n$. By assumption $s_n \to x$, so

$$\begin{aligned} \Rightarrow \alpha_i &= \langle s_n, e_i \rangle \to \langle x, e_i \rangle, \text{ for } i \leq k \\ \Rightarrow \alpha_i &= \langle x, e_i \rangle\ \forall\ i = 1, 2, \ldots \end{aligned}$$

3. This follows from Bessel's inequality and 1.

□

Definition (Total Set)**.** Let X be an inner product space and let $M \subset X$.

1. $\overline{\text{span}\,(M)} = X \Rightarrow M$ is a **total set**.

2. If M is an orthonormal set then M is a **total orthonormal set**.

Remark 3.17.

1. *A total orthonormal family in X is sometimes called an orthonormal basis for X. Note this is* **not equivalent to an algebraic basis** *unless X is finite dimensional.*

2. *In every nontrivial Hilbert space $H \neq \{0\}$ there is a total orthonormal set.*

Definition (Hilbert Dimension)**.** The Hilbert dimension of X is the cardinality of the smallest orthonormal set, i.e., if $\Lambda = \left\{ M \text{ s.t. } \overline{\text{span}\,(M)} = H \right\}$ then the Hilbert dimension is

$$\inf_{M \in \Lambda} |M| \,.$$

Problem 21*.* Let X be an inner product space and let $M \subset X$. Then

1. If M is total in X then
$$x \perp M \Rightarrow x = 0 \tag{7}$$

2. If X is complete, then (7) $\Rightarrow M$ is total in X.

Solution. 1. Let H be the completion of X. Then, X regarded as a subspace of H, is dense in H. By assumption, M is total in X, so span$M$ is dense in X, and dense in H. It follows that the orthogonal complement of M in H is $\{0\}$.

2. If x is a Hilbert space and $M^{\perp} = \{0\}$, then the Dense Set Lemma implies that M is total in X.

□

Problem 22*.* Show that an orthonormal set M in a Hilbert space H is total iff

$$\sum_k |\langle x, e_k \rangle|^2 = \|x\|^2 \,, \ \forall \ x \in M.$$

This is called Parseval's equality.

Solution. If M is not total, by Problem 21 there is a nonzero $x \perp M$ in H. Since $x \perp M$ we have 0 on the left-hand side of Parseval's Equality which is not equal to $\|x\|$. Hence if Parceval's Equality holds for all $x \in H$, then M must be total in H.

Conversely, assume M to be total in H. Consider any $x \in H$ and its nonzero Fourier coefficients arranged in a sequence, i.e., $\langle x, e_1 \rangle, \langle x, e_2 \rangle, ...,$. Define $y = \sum \langle x, e_k \rangle e_k$. It follows that $x - y \in M^{\perp}$. Since M is total, then $x = y$. □

Theorem 3.18 (Separable Hilbert Space)**.** *Let H be a Hilbert space.*

1. *If H is separable then every orthonormal set is countable.*

2. *If H contains an orthonormal sequence which is total then H is separable.*

3.1 Representation of Functionals on Hilbert Spaces

Theorem 3.19 (Riesz)**.** *Let H be a Hilbert space. Every bounded, linear functional on H can be represented in terms of the inner product on H, i.e.,*

$$f(x) = \langle x, z \rangle$$

where z is uniquely determined by f and

$$\|z\| = \|f\|.$$

Proof.

1. Existence of z. If $f = 0$ take $z = 0$. Otherwise assume $f \neq 0$. Then $\mathcal{N}(f) \neq H$ and $\mathcal{N}(f) \neq H \Rightarrow \mathcal{N}(f)^{\perp} \neq \{0\}$. Let $w \in \mathcal{N}(f)^{\perp}$ s.t. $w \neq 0$ and set

$$v = f(x)w - f(w)x, \; x \in H.$$

Then

$$\begin{aligned} \Rightarrow & \quad f(v) = f(x)f(w) - f(w)f(x) = 0 \\ \Rightarrow & \quad v \in \mathcal{N}(f). \end{aligned}$$

Since $w \perp \mathcal{N}(f)$ we have

$$\begin{aligned} 0 &= \langle v, w \rangle \\ &= \langle f(x)w - f(w)x, w \rangle \\ &= f(x) \langle w, w \rangle - f(w) \langle x, w \rangle \\ &= f(x) \|w\|^2 - f(w) \langle x, w \rangle. \end{aligned}$$

Then

$$\begin{aligned} \Rightarrow f(x) &= \frac{f(w)}{\|w\|^2} \langle x, w \rangle \\ \Rightarrow f(x) &= \left\langle x, \overline{\left(\frac{f(w)}{\|w\|^2}\right)} w \right\rangle. \end{aligned}$$

Then

$$z = \overline{\left(\frac{f(w)}{\|w\|^2}\right)} w.$$

2. Uniqueness of z. Suppose $f(x) = \langle x, z_1 \rangle = \langle x, z_2 \rangle$. Then

$$\Rightarrow \langle x, z_1 - z_2 \rangle = 0, \ \forall \ x \in H.$$

Choose $x = z_1 - z_2$. Then

$$\begin{aligned} \Rightarrow & \quad \langle z_1 - z_2, z_1 - z_2 \rangle = 0 \\ \Rightarrow & \quad z_1 = z_2. \end{aligned}$$

3. $\|f\| = \|z\|$. If $f = 0$ then $z = 0$ and $\|f\| = \|z\| = 0$. For $f \neq 0$, $z \neq 0$. Note

$$\begin{aligned} f(z) &= \langle z, z \rangle \\ &= \|z\|^2, \text{ and} \\ \|z\|^2 &\leq \|f\| \|z\| \\ \Rightarrow \|z\| &\leq \|f\|. \end{aligned}$$

Also

$$\begin{aligned} |f(x)| &= |\langle x, z \rangle| \\ &\leq \|x\| \|z\| \\ \Rightarrow \|f\| &\leq \|z\|. \end{aligned}$$

This yields

$$\|f\| = \|z\|.$$

□

Lemma 3.20 (Equality)**.** *Let X be an inner product space. Then*

$$\langle x, w \rangle = \langle y, w \rangle \ \forall \ w \in X \Rightarrow x = y.$$

In particular,

$$\langle x, w \rangle = 0 \ \forall \ w \in X \Rightarrow x = 0.$$

Definition (Sesquilinear Form)**.** Let X, Y be vector spaces over the same scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A **sesquilinear form** (or **sesquilinear functional**) h on $X \times Y$ is a mapping

$$h : X \times Y \to \mathbb{K}$$

such that

$$\forall \ x, x_1, x_2 \in X \text{ and } y, y_1, y_2 \in Y \text{ and } \alpha, \beta \in \mathbb{K}$$

1. $h(x_1 + x_2, y) = h(x_1, y) + h(x_2, y)$

2. $h(x, y_1 + y_2) = h(x, y_1) + h(x, y_2)$

3. $h(\alpha x, y) = \alpha h(x, y)$

4. $h(x, \beta y) = \overline{\beta} h(x, y)$

Remark 3.21.

1. *If X, Y are real ($\mathbb{K} = \mathbb{R}$), then*

$$h(x, \beta y) = \beta h(x, y)$$

and h is said to be **bilinear**.

2. *If X, Y are vector spaces and $\exists\ c \in \mathbb{R}$ s.t.*

$$|h(x, y)| \leq c \|x\| \|y\|, \ \forall\ x, y$$

then h is bounded and

$$\begin{aligned} \|h\| &= \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{|h(x, y)|}{\|x\| \|y\|} \\ &= \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |h(x, y)| \\ \Rightarrow |h(x, y)| &\leq \|h\| \|x\| \|y\| . \end{aligned}$$

Theorem 3.22 (Riesz Representation)**.** *Let H_1, H_2 be Hilbert spaces and*

$$h : H_1 \times H_2 \to \mathbb{K}$$

a bounded sesquilinear form. Then h has a representation

$$h(x, y) = \langle Sx, y \rangle$$

where $S : H_1 \to H_2$ is a bounded, linear operator. S is uniquely determined by h and

$$\|S\| = \|h\| .$$

Proof.

1. Existence of S. Consider $\overline{h(x, y)}$ which is linear in y. If we fix an x, then $\overline{h(x, y)}$ is a bounded, linear functional and we can use the Riesz theorem to find z and represent

$$\overline{h(x, y)} = \langle y, z \rangle ,$$

hence

$$h(x,y) = \langle z, y \rangle . \tag{8}$$

Note that z is unique, but it depends on $x \in H_1$. This means that (8) with variable x defines an operator $S : H_1 \to H_2$ given by

$$z = Sx,$$

and we write

$$h(x,y) = \langle z, y \rangle = \langle Sx, y \rangle .$$

We must show S is linear. Observe

$$\begin{aligned} \langle S(\alpha x_1 + \beta x_2), y \rangle &= h(\alpha x_1 + \beta x_2, y) \\ &= \alpha h(x_1, y) + \beta h(x_2, y) \\ &= \alpha \langle Sx_1, y \rangle + \beta \langle Sx_2, y \rangle , \ \forall \ y \in H_2 \\ \Rightarrow S(\alpha x_1 + \beta x_2) &= \alpha S x_1 + \beta S x_2. \end{aligned}$$

2. Uniqueness of S. If $h(x,y) = \langle Sx, y \rangle = \langle Tx, y \rangle$, then $S = T$ by the equality lemma.

3. Boundedness of S. Note first

$$\begin{aligned} \|h\| &= \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{|\langle Sx, y \rangle|}{\|x\| \, \|y\|} \\ &\geq \sup_{\substack{x \neq 0 \\ Sx \neq 0}} \frac{|\langle Sx, Sx \rangle|}{\|x\| \, \|Sx\|} \\ &= \sup_{x \neq 0} \frac{\|Sx\|}{\|x\|} \\ &= \|S\| \\ \Rightarrow \|h\| &\geq \|S\| . \end{aligned}$$

So S is bounded. Also

$$\begin{aligned} \|h\| &= \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{|\langle Sx, y \rangle|}{\|x\| \, \|y\|} \\ &\leq \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{\|Sx\| \, \|y\|}{\|x\| \, \|y\|} \\ &\leq \sup_{x \neq 0} \frac{\|Sx\|}{\|x\|} \\ &= \|S\| \\ \Rightarrow \|h\| &\leq \|S\| . \end{aligned}$$

This yields

$$\|S\| = \|h\| \,.$$

□

Problem 23. Let H be a Hilbert space. Show that H' (the dual of H) is a Hilbert space with inner product $\langle\cdot,\cdot\rangle_1$ defined by

$$\langle f_z, f_v\rangle_1 = \overline{\langle z, v\rangle} = \langle v, z\rangle \,,$$

where

$$\begin{aligned} f_z(x) &= \langle x, z\rangle \\ f_v(x) &= \langle x, v\rangle \end{aligned}$$

Solution. It is easy to verify that $\langle\cdot,\cdot\rangle_1$ is an inner product on $H^{'}$. Since $\|f_z\| = \|z\| = (\langle z, z\rangle)^{\frac{1}{2}} = (\langle f_z, f_z\rangle_1)^{\frac{1}{2}}$ the norm on $H^{'}$ is induced by the inner product $\langle\cdot,\cdot\rangle_1$. We know that the normed dual is always a Banach space. It follows that $H^{'}$ is complete, and therefore a Hilbert space. □

Problem 24. Show that any Hilbert space H is isomorphic with its second dual, i.e.,

$$H \cong (H')' = H''.$$

Solution. Let $T : H \to h^{''}$ by $z \to F_z$, where $F_z : H^{'} \to K$ is defined by $F_z(f) = \langle f, f_z\rangle_1$, where $\langle\cdot,\cdot\rangle_1$ and f_z are the same as in Problem 23. By repeated applications of the Riesz representation theorem we have that $F(f) = \langle f, g\rangle_1$ for some $g \in h^{'}$ and $g = f_z$ for some $z \in H$. It follows that T is surjective.

Suppose that $z, w \in H$ and $F_z = F_w$. Then $\langle f, f_z\rangle_1 = \langle f, f_w\rangle_1$ for all $f \in H^{'}$ and we get that $f_z = f_w$ and $z = w$. Therefore, T is injective.

It is easy to see that T is linear.

Also, by the Riesz representation theorem we have that $\|F_z\| = \|f_z\| = \|z\|$, i.e., T preserves norms, and inner products. It follows that T is an isomorphism. □

3.2 Hilbert Adjoint

Definition (Hilbert Adjoint T^*)**.** Let H_1, H_2 be Hilbert spaces and let $T : H_1 \to H_2$ be a bounded, linear operator. Then the adjoint is $T^* : H_2 \to H_1$ s.t.

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \,, \ \forall\ x \in H_1 \text{ and } x \in H_2.$$

Theorem 3.23 (Existence of the Hilbert Adjoint)**.**

1. *T^* exists.*

2. T^* *is unique.*

3. $\|T^*\| = \|T\|$.

Proof. Observe that $h(y, x) = \langle y, Tx\rangle$ is sesquilinear form on $H_2 \times H_1$.

$$\begin{aligned} \Rightarrow |h(x,y)| &\leq \|y\| \, \|Tx\| \\ &\leq \|T\| \, \|x\| \, \|y\| \\ \Rightarrow \|h\| &\leq \|T\| \, . \end{aligned}$$

Also

$$\begin{aligned} \|h\| &= \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{|\langle y, Tx\rangle|}{\|y\| \, \|x\|} \\ &\geq \sup_{\substack{x \neq 0 \\ Tx \neq 0}} \frac{|\langle Tx, Tx\rangle|}{\|Tx\| \, \|x\|} \\ &= \|T\| \\ \Rightarrow \|h\| &= \|T\| \, . \end{aligned}$$

Therefore h is a bounded sesquilinear form. Using a Riesz representation, $h(x, y) = \langle T^* y, x\rangle$ where T^* exists, is unique with norm

$$\|T^*\| = \|h\| = \|T\| \, .$$

Also

$$\begin{aligned} \Rightarrow \langle y, Tx\rangle &= \langle T^* y, x\rangle \\ \Rightarrow \langle Tx, y\rangle &= \langle x, T^* y\rangle \, . \end{aligned}$$

□

Lemma 3.24 (Zero Operator)**.** *Let* X, Y *be inner product spaces and* $Q : X \to Y$ *be bounded, linear operators. Then*

1. $Q = 0$ *iff* $\langle Qx, y\rangle = 0 \ \forall \ x \in X$ *and* $y \in Y$

2. *Let* X *be* **complex** *and* $Q : X \to X$*. Then*

$$\langle Qx, x\rangle = 0 \ \forall \ x \in X \Rightarrow Q = 0.$$

Proof.

1. $\Rightarrow$.

$$\begin{aligned} Q = 0 &\Rightarrow Qx = 0 \;\forall\; x \in X \\ &\Rightarrow \langle Qx, y\rangle = \langle 0, y\rangle = 0\,\langle w, y\rangle = 0. \end{aligned}$$

$\Leftarrow$.

$$\begin{aligned} \langle Qx, y\rangle = 0 \;\forall\; x, y &\Rightarrow Qx = 0 \;\forall\; x, y \\ &\Rightarrow Q = 0. \end{aligned}$$

2. Let $x, y \in X$, then $v = \alpha x + y \in X$. Then

$$\begin{aligned} \langle Qv, v\rangle &= \langle Qx, x\rangle = \langle Qy, y\rangle = 0 \\ &\Rightarrow \\ 0 &= \langle Q(\alpha x + y), \alpha x + y\rangle \\ &= |\alpha|^2 \langle Qx, x\rangle + \langle Qy, y\rangle + \alpha \langle Qx, y\rangle + \overline{\alpha}\langle Qy, x\rangle \\ &= \alpha \langle Qx, y\rangle + \overline{\alpha}\langle Qy, x\rangle . \end{aligned}$$

Now, take $\alpha = 1$ then $\alpha = i$ to obtain the two relations

$$\begin{aligned} \langle Qx, y\rangle + \langle Qy, x\rangle &= 0 \\ \langle Qx, y\rangle - \langle Qy, x\rangle &= 0. \end{aligned}$$

Then $\langle Qx, y\rangle = 0 \Rightarrow Q = 0$ by 1.

□

Remark 3.25. *In the previous lemma, 2 is not necessarily true if X is real.*

Theorem 3.26 (Properties of The Hilber Adjoint)**.** *Let H_1, H_2 be Hilbert spaces. Let $S, T : H_1 \to H_2$ be bounded, linear operators and let α be a scalar. Then*

1. $\langle T^* y, x\rangle = \langle y, Tx\rangle$
2. $(S + T)^* = S^* + T^*$
3. $(\alpha T)^* = \overline{\alpha} T^*$
4. $(T^*)^* = T$
5. $\|T^*T\| = \|TT^*\| = \|T\|^2$
6. $T^*T = 0$ *iff* $T = 0$

7. $(ST)^* = T^* S^*$, *assuming* $H_1 = H_2$.

Definition (Self-Adjoint, Unitary, Normal)**.** Let H be a Hilbert space and $T : H \to H$.

1. T is self-adjoint (or Hermitian) if $T^* = T$
2. T is unitary if T is bijection and $T^* = T^{-1}$
3. T is normal if $TT^* = T^*T$

Remark 3.27.

1. *If T is self-adjoint then $\langle Tx, y\rangle = \langle x, Ty\rangle$.*
2. *If T is self-adjoint or normal then T is normal.*

Example 3.28. Consider $\mathbb{C}^n$ with $\langle x, y\rangle = x^T\overline{y}$. Let $T : \mathbb{C}^n \to \mathbb{C}^n$. If we specify a basis for $\mathbb{C}^n$, we can represent T, T^* by matrices A, B. Then

$$\begin{aligned} \langle Tx, y\rangle &= (Ax)^T\overline{y} \\ &= x^T A^T \overline{y} \\ \langle x, T^*y\rangle &= x^T\overline{B}\overline{y}. \end{aligned}$$

Therefore

$$\begin{aligned} \Rightarrow A^T &= \overline{B} \\ \Rightarrow B &= \overline{A}^T. \end{aligned}$$

If T is self-adjoint then $A = \overline{A}^T$.

Theorem 3.29 (Self-Adjointness)**.** *Let H be a Hilbert space and let $T : H \to H$ be a bounded, linear operator. Then*

1. *If T is self-adjoint then $\langle Tx, x\rangle$ is real for all $x \in H$.*
2. *If H is complex and $\langle Tx, X\rangle$ is real for all $x \in H$ then T is self-adjoint.*

Proof.

1. If T is self-adjoint, then for all $x \in X$ we have

$$\overline{\langle Tx, x\rangle} = \langle x, Tx\rangle = \langle Tx, x\rangle .$$

2. If $\langle Tx, x\rangle$ is real for all $x \in X$, then

$$\langle Tx, x\rangle = \overline{\langle Tx, x\rangle} = \overline{\langle x, T^*x\rangle} = \langle T^*x, x\rangle .$$

   Hence

$$\begin{aligned} 0 &= \langle Tx, x\rangle - \langle T^*x, x\rangle \\ &= \langle (T - T^*)x, x\rangle \\ \Rightarrow T &= T^* \end{aligned}$$

   by the zero operator lemma, since H is complex.

□

Theorem 3.30. *Let H be a Hilbert space and let (T_n) be a sequence of bounded, linear, self-adjoint operators $T_n : H \to H$. Suppose that $T_n \to T$ in norm, that is $\|T_n - T\| \to 0$, where $\|\cdot\|$ is the norm on the space $B(H, H)$. Then T is a bounded, linear, self-adjoint operator on H.*

Proof. We show $T^* = T$.

$$\begin{aligned} \|T - T^*\| &\leq \|T - T_n\| + \|T_n - T_n^*\| + \|T_n^* - T^*\| \\ &= \|T - T_n\| + 0 + \|T_n^* - T^*\| \\ &= 2\,\|T - T_n\| \\ &\to 0. \end{aligned}$$

□

Theorem 3.31 (Unitary Operators)**.** *Let H be a Hilbert space and let $U, V : H \to H$ be unitary. Then*

1. *U is isometric, i.e., $\|Ux\| = \|x\|\ \forall\ x \in H$*
2. *$\|U\| = 1$ provided $H \neq \{0\}$*
3. *$U^{-1} = U^*$ is unitary*
4. *UV is unitary*
5. *U is normal.*
6. *If T is a bounded, linear operator on H and H is complex, then T is unitary iff T is isometric and surjective.*

Proof.

1.

$$\begin{aligned}\|Ux\|^2 &= \langle Ux, Ux\rangle \\ &= \langle x, U^*Ux\rangle \\ &= \langle x, Ix\rangle \\ &= \|x\|^2\end{aligned}$$

2. Follows from 1.

3. Since U is bijective, so is U^{-1} and

$$(U^{-1})^* = U^{**} = U = (U^{-1})^{-1}.$$

4. UV is bijective and

$$(UV)^* = V^*U^* = V^{-1}U^{-1} = (UV)^{-1}.$$

5. $U^{-1} = U^*$ and $UU^{-1} = U^{-1}U = I$.

6. $\Rightarrow$. Suppose that T is isometric and surjective. Isometry implies injectivity, so T is bijective. Need to show $T^* = T^{-1}$. By isometry,

$$\begin{aligned}\langle T^*Tx, x\rangle &= \langle Tx, Tx\rangle \\ &= \langle x, x\rangle \\ &= \langle Ix, x\rangle \\ \Rightarrow \langle (T^*T - I)x, x\rangle &= 0 \\ \Rightarrow T^*T &= I.\end{aligned}$$

Also,

$$\begin{aligned}TT^* &= TT^*(TT^{-1}) \\ &= T(T^*T)T^{-1} \\ &= I.\end{aligned}$$

Therefore $T^* = T^{-1}$.

$\Leftarrow$. Conversely, T is isometric by 1 and surjective by definition.

□

4 Fundamental Theorems

Definition (Partially Ordered Set)**.** Let M be a set with a relation $\leq$. This relation is said to be a partial order if

1. $a \leq a \;\forall\; a \in M$
2. $a \leq b$ and $b \leq a \Rightarrow a = b$
3. $a \leq b$ and $b \leq c \Rightarrow a \leq c$.

We say M is a partially ordered set.

Definition (Upper Bound)**.** Let M be partially ordered set. If $\exists\; u \in M$ s.t. $x \leq u \;\forall\; x \in M$, then u is said to be a upper bound. Such a u is not guaranteed to exist.

Definition (Maximal Element)**.** Let M be partially ordered set. If $\exists\; m \in M$ s.t. $m \leq x \Rightarrow m = x \;\forall\; x \in M$, then m is said to be a maximal elemtn. Such a m is not guaranteed to exist or be unique.

Example 4.1.

1. $\mathbb{R}$ with the usual $\leq$.
2. The power set $P(X)$ is partially ordered by set inclusion. X is the unique maximal element.
3. $x = (x_i), y = (y_i) \in \mathbb{R}^n$ ordered by
$$x \leq y \Leftrightarrow x_i \leq y_i \;\forall\; i = 1, \ldots, n.$$
4. The positive integers ordered by
$$x \leq y \Leftrightarrow x|n.$$

Definition (Chain)**.** Let M be a partially ordered set. A subset $C \subset M$ is said to be a chain if $a, b \in C \Rightarrow a \leq c$ or $c \leq a$. I.e., all elements in C are comparable.

The next statement is equivalent to the axiom of choice. It retain's its name (lemma) for historical reasons.

Lemma 4.2 (Zorn's Lemma)**.** *Let $M \neq \emptyset$ be a partially ordered set. Suppose that every chain $C \subset M$ has an upper bound. Then M has at least one upper bound.*

If we decide to live with the the axiom of choice, we can use Zorn's Lemma to prove various important results.

Theorem 4.3 (Hemel Basis)**.** *Every vector space $X \neq \{0\}$ has a Hemel basis.*

Proof. Let M be the set of all linearly independent subsets of X. Note that M is not empty:

$$X \neq \{0\} \Rightarrow\ \exists\, x \in X \text{ s.t. } x \neq 0 \Rightarrow \{x\} \in M.$$

Set inclusion gives a partial order on M. Let $C \subset M$ be a chain. Then

$$A = \bigcup_{D \in C} D$$

is an upper bound for C.

With this setup we invoke Zorn's lemma to force the existence of a maximal element B. Let $Y = \text{span}\,(B)$ and so $Y \subset X$. We want to show $Y = X$. Assume not and let $z \in X - Y$. Then $B \cup \{z\}$ is linearly independent, contradicting the maximality of B. □

Theorem 4.4 (Total Orthonormal Set)**.** *In ever Hilbert space $H \neq \{0\}$, there exists a total orthonormal set.*

Proof. Let M be the set of all orthonormal subsets of H. M is not empty since

$$X \neq \{0\} \Rightarrow\ \exists\, x \in X \text{ s.t. } x \neq 0 \Rightarrow \{\frac{x}{\|x\|}\} \in M.$$

Set inclusion gives a partial order on M. Let $C \subset M$ be a chain. Then

$$A = \bigcup_{D \in C} D$$

is an upper bound for C.

With this setup we invoke Zorn's lemma to force the existence of a maximal element F. We claim this F is total. To this end, suppose not. Then $\exists\, z \in H$ s.t. $z \neq 0$ and $z \perp F$. Hence $F \cup \{\frac{z}{\|z\|}\}$ is orthonormal, contradicting the maximality of F. □

Definition (Sublinear Functional)**.** Let X be a v.s. and let $p : X \to \mathbb{R}$. p is said to be a sublinear functional if

1. $p(x + y) \leq p(x) + p(y)\ \forall\ x, y \in X$
2. $p(\alpha x) = \alpha p(x)\ \forall\ x \in X\ \forall\ \alpha \in \mathbb{R}, \alpha \geq 0.$

Remark 4.5. *The norm on a n.s. is a sublinear functional.*

Theorem 4.6 (Hanh-Banach (Extensions of Linear Functionals))**.** *Let X be a real v.s. and let p be a sublinear functional on X. Let f be a linear functional defined on a subspace $Z \subset X$ s.t.*

$$f(x) \leq p(x)\ \forall\ x \in Z.$$

Then f has a linear extension $\hat{f} : Z \to X$ s.t.

$$\begin{aligned}\hat{f}(x) &\leq p(x)\ \forall\ x \in X \\ \hat{f}(x) &= f(x)\ \forall\ x \in Z.\end{aligned}$$

Proof.

1. Let E be the set of all linear extensions g of f which satisfy

$$g(x) \leq p(x)\ \forall\ x \in \mathcal{D}\left(g\right).$$

   Then $f \in E \Rightarrow E \neq \emptyset$. We now define the following partial order on E: $g \leq h$ means h is a linear extension of g, i.e.

   (a) $\mathcal{D}\left(g\right) \subset \mathcal{D}\left(h\right)$
   (b) $h(x) = g(x)\ \forall\ x \in \mathcal{D}\left(g\right)$.

   Let C be a chain in M. Then we define the maximal element $\hat{g}$ as follows

   (a) $\mathcal{D}\left(\hat{g}\right) = \bigcup_{g \in C} \mathcal{D}\left(g\right)$
   (b) $\hat{g}(x) = g(x)\ \forall\ g \in C$ s.t. $x \in \mathcal{D}\left(g\right)$.

   Then $g \leq \hat{g}\ \forall\ g \in C \Rightarrow \hat{g}$ is an upper bound for C. Since $C \subset E$ was arbitrary, we invoke Zorn's lemma to produce a maximal element $\hat{f}$. Therefore $\hat{f}$ is a linear extension of f.

2. Now we want to show $\mathcal{D}\left(\hat{f}\right) = X$. In the manner above, assume otherwise and let $y_1 \in x - \mathcal{D}\left(\hat{f}\right)$ and consider $Y_1 = \operatorname{span}\left(\{\}\mathcal{D}\left(\hat{f}\right) \cup \{y_1\}\right\}$. If $x \in Y_1$ then we can write $x = y + \alpha y_1, y \in \mathcal{D}\left(\hat{f}\right)$ (and this representation is unique). Then let us define a functional $g_1 : Y_1 \to \mathbb{R}$ as

$$g_1(y + \alpha y_1) = \hat{f}(y) + \alpha c$$

   where c is any real constant. Then g_1 is linear extension of f. If we can show

$$g_1(x) \leq p(x)\ \forall\ x \in \mathcal{D}\left(g_1\right)$$

   then $g_1 \in E$, which would contradict the maximality of $\hat{f}$ and imply $\mathcal{D}\left(\hat{f}\right) = X$.

3. To this end, let us construct a suitable c. Let $y, z \in \mathcal{D}\left(\hat{f}\right)$. Then

$$\begin{aligned}\hat{f}(y) - \hat{f}(z) &= \hat{f}(y - z) \\ &\leq p(y - z) \\ &= p(y + y_1 - y_1 - z) \\ &\leq p(y + y_1) + p(-y_1 - z)\end{aligned}$$

and therefor

$$-p(-y_1 - z) - \hat{f}(z) \leq p(y + y_1) - \hat{f}(y).$$

Note

(a) y_1 is fixed

(b) y does not appear on the RHS

(c) z does not appear on the LHS.

Then we can take $\sup_{z\in\mathcal{D}(\hat{f})}, \inf_{y\in\mathcal{D}(\hat{f})}$ and call these values m_0, m_1, respectively. Then $m_0 \leq m_1$. Let c be any value such that $m_0 \leq c \leq m_1$. Then

$$-p(-y_1 - z) - \hat{f}(z) \leq c \;\forall\; z \in \mathcal{D}\left(\hat{f}\right)$$
$$c \leq p(y + y_1) - \hat{f}(y) \;\forall\; y \in \mathcal{D}\left(\hat{f}\right).$$

Now let $\alpha \in \mathbb{R}$ and consider the cases

(a) $\alpha < 0$.

$$\begin{aligned} -p(-y_1 - \frac{1}{\alpha}y) - \hat{f}(\frac{1}{\alpha}y) &\leq c \\ \alpha p(-y_1 - \frac{1}{\alpha}y) + \hat{f}(y) &\leq -\alpha c \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} g_1(x) &= g(y + \alpha y_1) \\ &= \hat{f}(y) + \alpha c \\ &\leq -\alpha p(-y_1 - \frac{1}{\alpha}y) \\ &= p(\alpha y_1 + y) = p(x). \end{aligned}$$

(b) $\alpha = 0$. Then $x \in \mathcal{D}\left(\hat{f}\right)$ so $g_1(x) = \hat{f}(x) \leq p(x)$.

(c) $\alpha > 0$.

$$\begin{aligned} c &\leq p(\frac{1}{\alpha}y + y_1) - \hat{f}(\frac{1}{\alpha}y) \\ \alpha c &\leq \alpha p(\frac{1}{\alpha}y + y_1) - \hat{f}(y) \\ &= p(x) - \hat{f}(y) \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} g_1(x) &= \hat{f}(y) + \alpha c \\ &\leq p(x). \end{aligned}$$

Then $g_1 \in E$ and we have our contradiction. Therefore $\hat{f}$ is our bounded linear extension.

□

Problem 25.

1. Show that the norm on a v.s. X is a sublinear functional on X.
2. Show that a sublinear functional p satisfies $p(0) = 0$ and $p(-x) \geq -p(x)$.

Solution.

1. Let $p(x) = \|x\|$. Then $p(x+y) = \|x+y\| \leq \|x\| + \|y\| = p(x) + p(y)$. Also, if $\alpha \geq 0$, then
$$p(\alpha x) = \|\alpha x\| = |\alpha|\|x\| = \alpha\|x\| = \alpha p(x)$$
. It follows that p is a sublinear functional.
2. X is not empty, $0 \in X$. Let $x \in X$, then $p(0) = p(0x) = 0p(x) = 0$. Also, $0 = p(0) = p(x-x) \leq p(x) + p(-x)$. It follows that $p(-x) \geq -p(x)$.

□

Problem 26. If p is a sublinear functional on a real v.s. X, show that there exists a linear functional $\hat{f}$ on X s.t.
$$-p(-x) \leq \hat{f}(x) \leq p(x).$$

Solution. Let $Z = \{x \in X : x = \alpha x + 0, \alpha \in R\}$ and define the linear functional f on Z by

$$f(x) = \alpha p(x_0).$$

Then $f(x) \leq p(x)$. By the Hahn-Banach Theorem there exists a $\hat{f}$ bounded linear functional defined on X such that $\hat{f}(x) \leq p(x)$. Clearly, $-\hat{f}(x) = \hat{f}(-x) \leq p(x)$. It follows that $-p(-x) \leq \hat{f}(x) \leq p(x)$. □

Theorem 4.7 (Hanh-Banach Generalized)**.** *Let X be a v.s. over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and let p be a real-valued functional X s.t.*

1. $p(x+y) \leq p(x) + p(y) \ \forall \ x, y \in X$
2. $p(\alpha x) = |\alpha| \, p(x) \ \forall \ \alpha \in K$.

Let $Z \subset X$ be a subspace and let $f : Z \to K$ be a functional s.t.

$$|f(x)| \leq p(x) \ \forall \ x \in Z.$$

Then f has a bounded linear extension $\hat{f}$ s.t.

1. $\mathcal{D}\left(\hat{f}\right) = X$
2. $\hat{f}(x) = f(x) \ \forall \ x \in Z$
3. $\left|\hat{f}(x)\right| \leq p(x) \ \forall \ x \in X.$

Proof.

1. $\mathbb{K} = \mathbb{R}$. Then
$$|f(x)| \leq p(x) \Rightarrow f(x) \leq p(x).$$
Then by the previous theorem $\exists \ \hat{f}$ s.t. $\hat{f}(x) \leq p(x) \ \forall \ x \in X$. Also,
$$-\hat{f}(x) = \hat{f}(-x) \leq p(-x) = p(x)$$
so
$$\left|\hat{f}(x)\right| \leq p(x).$$

2. $\mathbb{K} = \mathbb{C}$. X complex $\Rightarrow Z$ complex $\Rightarrow f$ complex $\Rightarrow$
$$f = f_1 + i f_2$$
where f_1, f_2 are real-valued. We introduce X_r, Z_r the v.s.'s obtained by restricting $\mathbb{K}$ to $\mathbb{R}$. Now note that f linear on Z and f_1, f_2 real-valued means f_1, f_2 are real-valued linear functionals on Z_r. Also,
$$f_1(x) \leq |f(x)| \Rightarrow f_1(x) \leq p(x) \ \forall \ x \in Z_r.$$
Then by the previous theorems $\exists \ \hat{f}_1$ extension of f_1 from Z_r to X_r s.t.
$$\hat{f}_1(x) \leq p(x) \ \forall \ x \in X_r.$$
Going back to Z, we observe $\forall \ x \in Z$
$$\begin{aligned} i f(x) &= i\left[f_1(x) + i f_2(x)\right] \\ &= f_1(ix) + i f_2(ix) \\ &= -f_2(x) + i f_1(x) \\ \Rightarrow f_2(x) &= -f_1(ix) \end{aligned}$$

So then $\forall\, x \in X$ we define

$$\hat{f}(x) = \hat{f}_1(x) - i\hat{f}_1(ix).$$

Note

$$\hat{f}(x) = f(x) \;\forall\; x \in Z.$$

(a) We claim $\hat{f}$ is a linear functional on X.

$$\begin{aligned}
\hat{f}((a+ib)x) &= \hat{f}_1(ax+ibx) - i\hat{f}_1(iax - bx) \\
&= a\hat{f}_1(x) + b\hat{f}_1(ix) - i\left[a\hat{f}_1(ix) - b\hat{f}_1(x)\right] \\
&= (a+ib)\left[\hat{f}_1(x) - i\hat{f}_1(ix)\right] \\
&= (a+ib)\hat{f}(x)
\end{aligned}$$

(b) We claim $\left|\hat{f}(x)\right| \leq p(x) \;\forall\; x \in X$. Recall that $p(x) \geq 0$. Then

$$\hat{f}(x) = 0 \Rightarrow \left|\hat{f}(x)\right| \leq p(x).$$

Assume $\hat{f}(x) \neq 0$. We then use the exponential form of $\hat{f}$

$$\hat{f}(x) = \left|\hat{f}(x)\right| e^{i\theta}.$$

Then

$$\begin{aligned}
\left|\hat{f}(x)\right| &= \hat{f}(x)e^{-i\theta} \\
&= \hat{f}(e^{-i\theta}x) \\
&= \hat{f}_1(e^{-i\theta}x) + i\hat{f}_1(ie^{-i\theta}x) \\
\left|\hat{f}(x)\right| \in \mathbb{R} \Rightarrow & \\
\left|\hat{f}(x)\right| &= \hat{f}_1(e^{-i\theta}x) \\
&\leq p(e^{-i\theta}x) \\
&= \left|e^{-i\theta}\right| p(x) \\
&= p(x)
\end{aligned}$$

Then $\hat{f}$ is the bounded, linear extension.

$\square$

Theorem 4.8 (Hanh-Banach (for Normed Spaces)). *Let X be a n.s. and let $Z \subset X$ be a subspace. Let $f : Z \to \mathbb{K}$ be a bounded, linear functional. Then there exists a bounded, linear functional $\hat{f}$ which is an extension from Z to X s.t.*

$$\left\|\hat{f}\right\|_X = \|f\|_Z$$

where

$$\begin{aligned} \left\|\hat{f}\right\|_X &= \sup_{\substack{x\in X\\ \|x\|=1}} \left|\hat{f}(x)\right| \\ \|f\|_Z &= \sup_{\substack{x\in Z\\ \|x\|=1}} |f(x)|. \end{aligned}$$

Proof. If $Z = \{0\}$ then $f = 0$ and $\hat{f} = 0$. Otherwise, let $Z \neq \{0\}$. Then $\forall\, x \in Z$ we have

$$|f(x)| \leq \|f\|_Z \|x\|.$$

Then define

$$p(x) \leq \|f\|_Z \|x\| \;\; \forall\, x \in X.$$

Notice

$$\begin{aligned} p(x+y) &= \|f\|_Z \|x+y\| \\ &\leq \|f\|_Z (\|x\| + \|y\| \\ &= p(x) + p(y) \\ p(\alpha x) &= \|f\|_Z \|\alpha x\| \\ &= |\alpha| \, \|f\|_Z \|x\| \\ &= |\alpha| \, p(x). \end{aligned}$$

Then we can use the Hanh-Banach (Generalized) theorem $\Rightarrow \;\exists\, \hat{f} : X \to \mathbb{K}$ s.t.

$$\left|\hat{f}(x)\right| \leq p(x) = \|f\|_Z \|x\| \;\; \forall\, x \in X.$$

And so

$$\left\|\hat{f}\right\|_X = \sup_{\substack{x\in X\\ \|x\|=1}} \left|\hat{f}(x)\right| \leq \|f\|_Z.$$

Finally, since $\hat{f}$ is an extension of f

$$\left\|\hat{f}\right\|_X \leq \|f\|_Z.$$

Therefore,

$$\left\|\hat{f}\right\|_X = \|f\|_Z.$$

□

Remark 4.9. *In the previous theorem if $X = H$ is a Hilbert space and $Z \subset H$ is a closed subspace then we can represent f (for some fixed $z \in Z$)*

$$\begin{aligned} f(x) &= \langle x, z\rangle \ \forall\ x \in Z \\ \|f\| &= \|z\| \\ \hat{f}(x) &= \langle x, z\rangle \ \forall\ x \in X \end{aligned}$$

Theorem 4.10 (Bounded Linear Functionals)**.** *Let X be a n.s. and let $x_0 \in X, x_0 \neq 0$. Then $\exists\ \hat{f} : X \to \mathbb{K}$ s.t.*

$$\begin{aligned} \left\|\hat{f}\right\| &= 1 \\ \hat{f}(x_0) &= \|x_0\| \end{aligned}$$

Proof. Let $Z \subset X$ be the subspace defined as

$$Z = \{x \in X | x = \alpha x_0\}.$$

Then

$$\begin{aligned} f(x) &= f(\alpha x_0) \\ &= \alpha \|x_0\| \\ |f(x)| &= |f(\alpha x_0)| \\ &= |\alpha| \|x_0\| \\ &= \|\alpha x_0\| \\ &= \|x\| \\ \Rightarrow \|f\| &= 1. \end{aligned}$$

Therefore f has a linear extension $\hat{f}$ from Z to X with norm

$$\left\|\hat{f}\right\| = \|f\| = 1.$$

By definition of f and $\hat{f}$, $\hat{f}(x_0) = f(x_0) = \|x_0\|$. □

Problem 27. To illustrate the Hanh-Banach (for Normed Spaces) consider a functional $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x) = \alpha_1 x_1 + \alpha_2 x_2 \ \forall\ x = (x_1, x_2) \in \mathbb{R}^2.$$

Construct its linear extension to $\mathbb{R}^3$ and the corresponding norms.

Solution. Let $\hat{f}(x) = \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3$. Then $\|\hat{f}\|(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)^{\frac{1}{2}} \geq \|f\|$ and $\|\hat{f}\| = \|f\|$ iff $\alpha_3 = 0$. □

Problem 28. Consider the n.s. $\mathbb{R}^2$ and let $x_0 \in \mathbb{R}^2, x_0 \neq 0$. Find a bounded, linear functional $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$ s.t.

1. $\left\|\hat{f}\right\| = 1$
2. $\hat{f}(x_0) = \|x_0\|$

Solution. let $\hat{f} = \langle x, \frac{x_0}{\|x_0\|}\rangle$. Then $\|\hat{f}\|\|\frac{x_0}{\|x_0\|}\| = 1$ and $\hat{f}(x_0) = \langle x_0, \frac{x_0}{\|x_0\|}\rangle = \|x_0\|$. □

4.1 Bounded, Linear Functionals on $C[a, b]$

Definition (Variation)**.** Let $f : [a, b] \to \mathbb{R}$. We define the variation of f as

$$\operatorname{Var} f = \sup\left\{\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|\right\}$$

where

$$a = t_0 < t_1 < \cdots < t_n = b$$

is a partition of $[a, b]$.

Definition (Bounded Variation)**.** We say f is of bounded variation and write $f \in BV[a, b]$ if

$$\operatorname{Var} f < \infty.$$

Note $BV[a, b]$ is a vector space. If we define the norm

$$\|f\| = |f(a)| + \operatorname{Var} f,\ \forall\ f \in BV[a, b]$$

then $BV[a, b]$ is a normed space.

Definition (Riemann-Stieltjes Integral)**.** Let $x \in C[a, b]$ and $f \in BV[a, b]$ and P_n be a partition of $[a, b]$. Define

$$\eta P_n = \max\left\{t_1 - t_0, \ldots, t_n - t_{n-1}\right\}.$$

Consider S defined by

$$S(P_n) = \sum_{i=1}^{n} x(t_i)\left[f(t_i) - f(t_{i-1})\right].$$

Then $\exists\ I \in \mathbb{R}$ s.t. $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ s.t.

$$\eta P_n < \delta \Rightarrow |I - S(P_n)| \leq \varepsilon.$$

We call I the Riemann-Stieltjes integral of x with respect to f and write

$$I = \int_a^b x(t)df(t).$$

Remark 4.11.

1. *If $f(t) = t$ then we have the usual integral.*

2. *If f has a integrable derivative then*

$$\int_a^b x(t)df(t) = \int_a^b x(t)f'(t)dt.$$

3. *We have*

$$\left|\int_a^b x(t)df(t)\right| \leq \sup_{t\in[a,b]} |x(t)| \operatorname{Var}(f).$$

Theorem 4.12 (Reisz - B.L.F.'s on $C[a,b]$)**.** *Every b.l.f. g on $C[a,b]$ can be represented by a Riemann-Stieltjes integral*

$$g(x) = \int_a^b x(t)df(t)$$

and

$$\|g\| = \operatorname{Var}(f).$$

Proof. By the Hanh-Banach theorem g has an extension from $C[a,b]$ to $B[a,b]$ the space of bounded functions with norm

$$\|x\| = \sum_{t\in[a,b]} |x(t)|$$

and

$$\|\hat{g}\| = \|g\|.$$

Let

$$x_t(s) = \begin{cases} 1 & s \in [0,t] \\ 0 & otherwise \end{cases},$$

then $x_t \in B[a,b]$. Let $f(a) = 0$ and $f(t) = \hat{g}(x_t)\ \forall\ t \in (a,b]$. We want to show f is of b.v. and $\operatorname{Var} F \leq \|g\|$.

Use exponential form for a complex quantity z, $z = |z|\, e(z)$ where

$$e(z) = \begin{cases} 1 & z = 0 \\ e^{i\theta} & z \neq 0 \end{cases}.$$

Note that if $z \neq 0$ then $|z| = ze^{-i\theta} \Rightarrow$

$$|z| = z\overline{e(z)}.$$

We shall use the notation

$$\varepsilon_i = \overline{e(f(t_i) - f(t_{i-1}))} \text{ and } x_{t_i} = x_i.$$

Then

$$\begin{aligned}
\sum_{i=1}^{n} [f(t_i) - f(t_{i-1})] &= |\hat{g}(x_1)| + \sum_{i=2}^{n} |\hat{g}(x_i) - \hat{g}(x_{i-1})| \\
&= \varepsilon_1 \hat{g}(x_1) + \sum_{i=1}^{n} \varepsilon_i [\hat{g}(x_i) - \hat{g}(x_{i-1})] \\
&= \hat{g}\left(\varepsilon_1 x_1 + \sum_{i=2}^{n} \varepsilon_i [x_i - x_{i-1}]\right) \\
&\leq \|\hat{g}\| \left\| \varepsilon_1 x_1 + \sum_{i=2}^{n} \varepsilon_i [x_i - x_{i-1}] \right\| \\
&= \|g\| \cdot 1 \\
\Rightarrow \operatorname{Var} f &\leq \|g\| \\
\Rightarrow f &\in BV[a,b]
\end{aligned}$$

Now, given P a partition of $[a,b]$ define

$$z_n = x(t_0)x_1 + \sum_{i=2}^{n} x(t_{i-1} [x_i - x_{i-1}]$$

then $z_n \in BV[a,b]$. Now we compute

$$\begin{aligned}
\hat{g}(z_n) &= x(t_0)\hat{g}(x_1) + \sum_{i=2}^{n} x(t_{i-1}) [\hat{g}(x_i) - \hat{g}(x_{i-1})] \\
&= x(t_0)f(x_1) + \sum_{i=2}^{n} x(t_{i-1}) [f(x_i) - f(x_{i-1})] \\
&= \sum_{i=1}^{n} x(t_{i-1}) [f(x_i) - f(x_{i-1})]
\end{aligned}$$

Then taking a sequence of partitions (P_n) s.t. $\eta P_n \to 0$, we obtain

$$\int_a^b x(t) df(t).$$

We need to show $\hat{g}(z_n) \to \hat{g}(x) = g(x)$ where $x \in C[a,b]$. Note $z_n(a) = x(a) \cdot 1 \Rightarrow z_n(z) = x(a) = 0$. Note

$$t_{i-1} < t \leq t_i \Rightarrow |z_n(t) - x(t)| = |x(t_{i-1}) - x(t)|.$$

Note

$$\eta P_n \to 0 \Rightarrow \|z_n - x\| \to 0$$

since $x \in C[a,b] \Rightarrow x$ is uniformly continuous (since $[a,b]$ is compact). Then the continuity of $\hat{g}$ implies $\hat{g}(z_n) \to \hat{g}(x)$ and $\hat{g}(x) = f(x)$.

Now we want to show $\operatorname{Var} f = \|g\|$. Computing,

$$|g(x)| \leq \max_{t\in[a,b]} x(t) \operatorname{Var} f = \|x\| \operatorname{Var} f$$

and so

$$\|g\| \leq \operatorname{Var} f.$$

Also $\operatorname{Var} f \leq \|g\|$, so

$$\|g\| = \operatorname{Var} f.$$

□

Remark 4.13. *f is not unique in the above theorem. However, we can make f unique by requiring:*

1. $f(a) = 0$
2. $f(t^+) = f(t)$ *(continuity from the right).*

4.2 Adjoint Operator

Given a b.l.o. $T : X \to Y$ we are interested in constructing a new operator $T^\times : Y' \to X'$. We proceed as follows. Let $g \in Y', x \in X$. Setting $y = Tx$ we obtain a function f on X by defining

$$f(x) = g(Tx).$$

f is linear because g, T are linear. f is bounded because

$$\begin{aligned} |f(x)| &= |g(Tx)| \leq \|g\| \|T\| \|x\| \\ \Rightarrow \|f\| &\leq \|g\| \|T\| . \end{aligned}$$

Therefor $f \in X'$. Then

$$f(x) = g(Tx) \tag{9}$$

with variable $g \in Y'$ defines an operator called the adjoint of T,

$$T^\times : Y' \to X'.$$

Definition (Adjoint)**.** Given a b.l.o. $T : X \to Y$ the adjoint $T^\times : Y' \to X'$ is given by

$$f(x) = (T^\times g)(x) = g(Tx).$$

Theorem 4.14. $\|T^\times\| = \|T\|$.

Proof. We know $T^\times$ is linear and $f = T^\times g$. Then

$$\|T^\times g\| = \|f\| \leq \|g\| \, \|T\| \Rightarrow \|T^\times\| \leq \|T\| .$$

Now we want to show $\|T^\times\| \geq \|T\|$. For any $x_0 \in X, x_0 \neq 0 \; \exists \; g_0 \in Y'$ s.t. $\|g_0\| = 1$ and

$$g_0(Tx_0) = \|Tx_0\| .$$

Hence

$$\begin{aligned} g_0(Tx_0) &= (T^\times g_0)(x_0) \\ f_0 &= T^\times g_0 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \|Tx_0\| &= g(Tx_0) \\ &= f_0(x_0) \\ &\leq \|f_0\| \, \|x_0\| \\ &= \|T^\times g_0\| \, \|x_0\| \\ &\leq \|T^\times\| \, \|g_0\| \, \|x_0\| . \end{aligned}$$

Recall $\|g_0\| = 1$, then

$$\|Tx_0\| \leq \|T^\times\| \, \|x_0\| \Rightarrow \|T^\times\| \geq \frac{\|Tx_0\|}{\|x_0\|}.$$

But $x_0 \neq 0$ is arbitrary, and taking sup on the RHS yields

$$\|T^\times\| \geq \|T\| .$$

□

Example 4.15. Let $T : \mathbb{R}^n \to \mathbb{R}^n$. Let E be a basis, and let $x, y \in \mathbb{R}^n$, and let T_E be the representation of T with respect to E.

$$\begin{aligned} E &= \{e_1, \ldots, e_n\} \\ x &= (\xi_1, \ldots, \xi_n) \\ y &= (\eta_1, \ldots, \eta_n) \\ T_E &= (\tau_{ik}) \\ y &= T_E x \\ \eta_i &= \sum_{k=1}^{n} \tau_{ik} \xi_k . \end{aligned}$$

Let $F = \{f_1, \ldots, f_n\}$ be the dual basis of E, i.e., a basis for $\mathbb{R}^{n\prime}$. Then for $g \in \mathbb{R}^{n\prime}$, we can represent

$$\begin{aligned}
g &= \sum_{i=1}^{n} \alpha_i f_i \\
f_i(y) &= f_i\left(\sum_{i=1}^{n} \eta_k e_k\right) = \eta_i \\
\Rightarrow g(y) &= g\left(T_E x\right) \\
&= \sum_{i=1}^{n} \alpha_i \eta_i \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_i \tau_{ik} \xi_k \\
\Rightarrow g\left(T_E x\right) &= \sum_{k=1}^{n} \beta_k \xi_k \text{ where } \beta_k = \sum_{i=1}^{n} \tau_{ik} \alpha_i \\
\Rightarrow f(x) &= g\left(T_E x\right) \\
&= \sum_{k=1}^{n} \beta_k \xi_k \\
\Rightarrow f &= \left(T_E\right)^{\times} g \\
\beta_k &= \sum_{i=1}^{n} \tau_{ik} \alpha_i .
\end{aligned}$$

Therefor, if T is represented by T_E, then $T^{\times}$ is represented by the transpose of T_E.

Remark 4.16 (Properties of Adjoint)**.**

1. $(S+T)^{\times} = S^{\times} + T^{\times}$
2. $(\alpha T)^{\times} = \alpha T^{\times}$
3. $(ST)^{\times} = T^{\times} S^{\times}$
4. *If* $T \in B(X, Y)$ *and* T^{-1} *exists and* $T^{-1} \in B(Y, X)$*, then* $(T^{\times})^{-1}$ *also exists,* $(T^{\times})^{-1} \in B(X', Y')$ *and* $(T^{\times})^{-1} = (T^{-1})^{\times}$*.*

Remark 4.17 (Reltaion between $T^{\times}$ and T^{*})**.** *Let* $T : X \to Y, X = H_1, Y = H_2, T : H_1 \to H_2, T^{\times} : H_2' \to H_1', f \in H_1' g \in H_2'$

$$\begin{aligned}
T^{\times} g &= f \\
g(Tx) &= f(x)
\end{aligned}$$

where H_1, H_2 are Hilbert spaces. Then let us represnet (Riesz)

$$\begin{aligned} f(x) &= \langle x, x_0 \rangle, \quad x_0 \in H_1 \\ g(y) &= \langle y, y_0 \rangle, \quad y_0 \in H_2. \end{aligned}$$

Then we can define $A_1 : H_1' \to H_1$ by $A_1 f = x_0$ and $A_2 : H_2' \to H_2$ by $A_2 g = y_0$. Then A_1, A_2 are bijective, isometric, and conjugate-linear. So we can define $T^* : H_2 \to H_1$ by

$$T^* y_0 = A_1 T^\times A_2^{-1} y_0 = x_0.$$

$\Rightarrow$

$$\begin{aligned} \langle Tx, y_0 \rangle &= g(Tx) \\ &= f(x) \\ &= \langle x, x_0 \rangle \\ &= \langle x, T^* y_0 \rangle \end{aligned}$$

Remark 4.18.

1. *$(\alpha T)^\times = \alpha T^\times$ but $(\alpha T)^* = \overline{\alpha} T^*$*
2. *In the finite dimensional setting T^* is represented as the transpose and T^* is represented by the conjugate transpose:*

$$\begin{aligned} T^\times &= T^T \\ T^* &= \overline{T}^T. \end{aligned}$$

Problem 29.

1. Show $(\alpha T)^\times = \alpha T^\times$.
2. Show $(T^n)^\times = (T^\times)^n$.

Solution.

1. $((\alpha T)^x g)(x) = g((\alpha T)x) = g(\alpha Tx) = \alpha g(Tx) = \alpha(t^x g)(x) = ((\alpha T^x)g)(x)$.
2. We have $(ST)^x = T^x S^x$ because $((ST)^x g)(x) = g((ST)(x)) = g(S(T(x))) = (s^x g)(Tx) = T^x((S^x g)(x)) = (T^x(S^x g))(x) = ((T^x S^x)g)(x)$. It follows that $(T^n)^x = (T^{n-1})^x T^x = \cdots = (t^x)^n$.

□

4.3 Reflexive Spaces

Definition (Reflexive (Algebraic))**.** A vector space X is algebraically reflexive if the canonical map $C : X \to X^{**}$ is surjective (and thus bijective). Recall C is the mapping

$$x \mapsto g_x \text{ where } g_x(f) = f(x),\ \forall\ f \in X^*.$$

Definition (Reflexive (Dual))**.** A normed space X is called **reflexive** if $X = X''$, i.e., if $\mathcal{R}(C) = X''$.

Problem 30. Let X be a n.s.. Define $g_x : X' \to \mathbb{K}$ by fixing an $x \in X$ and setting

$$g_x(f) = f(x)\ \forall\ f \in X'.$$

1. Show $\forall$ fixed $x \in X, g_x$ is a b.l.f. on X' and
$$\|g_x\| = \|x\|.$$
2. C is an isomorphism $X \overset{\text{onto}}{\longrightarrow} \mathcal{R}(C)$.
3. If a n.s. X is reflexive (i.e., $\mathcal{R}(C) = X''$) then X is complete.

Solution.

1. Let $g_x(f) = f(x)$, where $x \in X$ is fixed. Then $|g_x(f)| = |f(x)| \leq \|f\|\|x\|$ and therefore g is bounded and $\|g\| \leq \|x\|$. Let $\hat{f} \in X^{'}$ such that $\|\hat{f}\| = 1$ and $\hat{f}(x) = \|x\|$. (We established the existence of such $\hat{f}$.) Then $\|g_x\| \geq \frac{|g_x(\hat{f})|}{\|\hat{f}\|} = \|x\|$. It follows that $\|g_x\| = \|x\|$.
2. Consider $C : x \to X^{''}$, where $x \to g_x$. Then C is linear, because $\forall f \in X^{'}$ we have
$$g_{\alpha x+\beta y}(f) = f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) = \alpha g_x(f) + \beta g_y(f).$$
Then $C(\alpha x + \beta y) = \alpha g_x + \beta g_y = \alpha C(x) + \beta C(y)$. Also, $\|C(x)\| = \|g_x\| = \|x\|$, and then $\|g_x - g_y\| = \|g_{x-y}\| = \|x - y\|$ and C is isometric. Moreover, if $x \neq y$, then $g_x \neq g_y$ and therefore C is injective. It follows that C is an isomorphism onto its range.
3. $X^{''}$ is complete being the dual of $X^{'}$. By assumption X is reflexive, hence $R(C) = X^{''}$. Part (ii) implies the completeness of X via isomorphism.

□

Theorem 4.19. *Every f.d.n.s. is reflexive.*

Example 4.20.

1. $\ell^p, \mathcal{L}^p[a,b],\ \forall\ 1 < p < \infty$ are reflexive.

2. $C[a,b], \mathcal{L}^1[a,b], \ell^1$ are not reflexive.

Theorem 4.21 (Hilbert Spaces are Reflexive)**.** *If H is a Hilbert space then H is reflexive.*

Proof. The canonical map $C : H \to H''$ given by $x \mapsto g_x$ is injective. We want to show C is surjective, i.e.,

$$\forall\ g \in H''\ \exists\ x \in H \text{ s.t. } g = Cx.$$

Define $A : H' \to H$ by by $Af = z$ where z is the Riesz representation of f:

$$f(x) = \langle x, z \rangle .$$

Then A is bijective, an isometry and conjugate-linear. We view H' as the Hilbert space with inner product

$$\langle f_1, f_2 \rangle_1 = \langle Af_2, Af_1 \rangle .$$

Let $g \in H''$ be arbitrary. Then

$$g(f) = \langle f, f_0 \rangle_1 = \langle Af_0, Af \rangle .$$

Writing $f(x) = \langle x, z \rangle, z = Af, Af_0 = z$ we have

$$\langle Af_0, Af \rangle = \langle x, z \rangle = f(x).$$

$\Rightarrow g(f) = f(x) \Rightarrow g = Cx$. Then H is reflexive. $\square$

Lemma 4.22 (Existence of a Functional)**.** *Let Y be a n.s. and let $Y \subset X$ be a proper, closed subspace. Let $x_0 \in X - Y$ be arbitrary. Calculate*

$$\delta = \inf_{\hat{y} \in Y} \|\hat{y} - x_0\| .$$

Note Y closed $\Rightarrow \delta > 0$. Then $\exists\ \hat{f} \in X'$ s.t.

$$\left\|\hat{f}\right\| = 1 \text{ and } \hat{f}(y) = 0\ \forall\ y \in Y \text{ and } \hat{f}(x_0) = \delta. \tag{10}$$

Proof. Consider a subspace $Z \subset X$ spanned by Y and x_0 and define a sublinear functional f on Z by

$$f(z) = f(y + \alpha x_0) = \alpha\delta.$$

Note $\delta \neq 0 \Rightarrow f \neq 0$. Then f satisfies (10). Now we use the Hanh-Banach theorem to extend f to X. In a slight abuse of notation we refer to the extension of f as f. Then f is bounded. It is clear that $\alpha = 0 \Rightarrow f(z) = 0$.

For the case $\alpha \neq 0$

$$\begin{aligned}
|f(z)| &= |\alpha|\,\delta \\
&= |\alpha| \inf_{\hat{y}\in Y} \|\hat{y} - x_0\| \\
&\leq |\alpha|\,\|-\alpha^{-1}y - x_0\| \\
&= \|y + \alpha x_0\| \\
\Rightarrow |f(z)| &\leq \|z\| \\
\Rightarrow \|f\| &\leq 1
\end{aligned}$$

Now to show $\|f\| \geq 1$. $\exists\, (y_n) \subset Y$ s.t. $\|y_n - x_0\| \to \delta$. Let $z_n = y_n - x_0$, then $f(z_n) = -\delta$. Then

$$\begin{aligned}
\|f\| &= \sup_{\substack{z\in Z\\ z\neq 0}} \frac{|f(z)|}{\|z\|} \\
&\geq \frac{|f(z_n)|}{\|z_n\|} \\
&= \frac{\delta}{\|z_n\|} \\
&\to 1,
\end{aligned}$$

$\Rightarrow \|f\| = 1.$ □

Theorem 4.23 (Separability)**.** *X' separable $\Rightarrow X$ separable.*

Proof. Assume X' is separable. Then

$$U' = \{f \,|\, \|f\| = 1\} \subset X'$$

contains a countable, dense subset, say $(f_n) \subset U'$. Since $f_n \in U'\ \forall\ n$,

$$\|f_n\| = \sup_{\|x\|=1} |f(x)| = 1.$$

By the definition of sup we can find a sequence $(x_n) \in X$ s.t. $\|x_n\| = 1\ \forall\ n$ s.t.

$$|f_n(x_n)| \geq \frac{1}{2}.$$

Let $Y = \overline{\operatorname{span}((x_n))}$, then Y is separable. We want to show $Y = X$ (because then (x_n) is a countable dense subset of X).

Suppose $Y \subsetneq X$. Then $\exists\ \hat{f} \in X'$ s.t. $\left\|\hat{f}\right\| = 1, \hat{f}(y) = 0\ \forall\ y \in Y$. Since $(x_n) \subset Y$ we have $\hat{f}(x_n) = 0\ \forall\ n$. Then

$$\begin{aligned}
\frac{1}{2} &\leq |f_n(x_n)| \\
&= \left|f_n(x_n) = \hat{f}(x_n)\right| \\
&= \left|(f_n - \hat{f})(x_n)\right| \\
&\leq \left\|f_n - \hat{f}\right\| \|x_n\| \\
&= \left\|f_n - \hat{f}\right\|.
\end{aligned}$$

Then $\left\|f_n - \hat{f}\right\| \geq \frac{1}{2}$ and $\left\|\hat{f}\right\| = 1$ contradicts that $(f_n) \subset U'$ is dense. ☐

Corollary 4.24. *X seprable and X' not separable $\Rightarrow X$ not reflexive.*

Proof. X reflexive $\Rightarrow X = X'' \Rightarrow X''$ separable $\Rightarrow X'$ separable, which contradicts our assumptions. ☐

Example 4.25. ℓ^1 is not reflexive. This is because $(\ell^1)' = \ell^\infty$ and ℓ^∞ is not separable.

4.4 Baire Category Theorem

Definition (Category)**.** Let X be a metric space and $M \subset X$.

1. M is said to be **nowhere dense** in X if $\overline{M}$ has no interior points.
2. M is said to be of **first category** or **meager** if X is the countable union of nowhere dense sets.
3. M is of the **second category** if it is not of first category.

Theorem 4.26 (Baire's Theorem)**.** *If $X \neq \emptyset$ then X is of the second category (i.e., X is not meager).*

Proof. Suppose $X \neq \emptyset$ and X is meager. Then

$$X \bigcup_{k=1}^{\infty} M_k$$

with M_k nowhere dense in $X\ \forall\ k$. By assumption, $\overline{M_1}$ does not contain a nonempty open set $\Rightarrow \overline{M_1} \neq X$, $\Rightarrow \overline{M_1}^c = X - \overline{M_1}$ is not empty and is open. Pick $p_1 \in \overline{M_1}^c$ and an open ball containing p_1, say $B_1 = B(p_1, \varepsilon_1) \subset \overline{M_1}^c$, with $\varepsilon_1 \leq \frac{1}{2}$.

Similarly, $\overline{M_2}$ does not contain a nonempty open set $\Rightarrow B(p_1, \frac{\varepsilon_1}{2}) \subsetneq \overline{M_2}$. Then $\overline{M_2}^c \cap B(p_1, \frac{\varepsilon_1}{2}) \neq \emptyset$. Then

$$\exists\ B_2 = B(p_2, \varepsilon_2) \subset \left(\overline{M_2}^c \cap B(p_1, \frac{\varepsilon_1}{2})\right) \text{ with } \varepsilon_2 < \frac{\varepsilon_1}{2}.$$

Continuing inductively, we can find $B_k = B(p_k, \varepsilon_k)$ with $\varepsilon_k < \frac{1}{2^k}$ s.t.

$$\begin{aligned} B_k \cap M_k &\neq \emptyset \\ B_{k+1} &\subset B(p_k, \frac{\varepsilon_k}{2}) \subset B_k. \end{aligned}$$

Since $\varepsilon < \frac{1}{2^k}$ we find (p_n) is Cauchy. Then because X is complete $\exists\ p \in X$ s.t. $p_n \to p$. Also $\forall\ n > m$

$$\begin{aligned} d(p_m, p) &\leq d(p_m, p_n) + d(p_n, p) \\ &< \frac{\varepsilon_m}{2} + d(p_n, p) \\ &\to \frac{\varepsilon_m}{2} \end{aligned}$$

and therefore $p \in B_m\ \forall\ m$. Since $B_m \subset \overline{M_m}^c$, we have $p \notin M_m\ \forall\ m$. Therefore $p \notin \cup M_m = X$, a contradiction. ☐

Theorem 4.27 (Uniform Boundedness)**.** *Let $(T_n) \subset B(X, Y)$ with X a Banach space and Y a normed space. Suppose $\forall\ x \in X\ \exists\ c_x$ s.t.*

$$\|T_n x\| \leq c_x\ \forall\ n.$$

Then $(\|T_n\|)$ is bounded.

Proof. For each k define

$$A_k = \{x \in X |\, \|T_n x\| \leq k\ \forall\ n\}.$$

Then we can see A_k is closed:

1. Let $x \in \overline{A_k}$. Then $\exists\ (x_i) \subset A_k$ s.t. $x_i \to x$.
2. For fixed n, $\|T_n x_i\| \leq k \Rightarrow \|T_n x\| \leq k$ by continuity of $\|\cdot\|$.
3. Then $x \in A_k$.

By assumption $\forall\ x \in X\ \exists\ k$ s.t. $x \in A_k$, so

$$X = \bigcup_{k=1}^{\infty} A_k.$$

Then since X is complete we invoke Baire to obtain A_{k_0}, a set which contains an open ball, say

$$B_0 = B(x_0, v) \subset A_{k_0}.$$

Let $x \in X$ s.t. $x \neq 0$ be arbitrary. Set $z = x_0 + \gamma x, \gamma = \frac{v}{2\|x\|}$. Then

$$\begin{aligned} \|z - x_0\| < r &\Rightarrow z \in B_0 \\ &\Rightarrow \|T_n z\| \leq k \ \forall \ n \end{aligned}$$

Also, $x_0 \in B_0 \Rightarrow \|T_n x_0\| \leq k_0$. Then

$$\begin{aligned} \|T_n x\| &= \gamma^{-1} \|T_n(z - x_0)\| \\ &\leq \gamma^{-1} (\|T_n z\| + \|T_n x_0\|) \\ &\leq \frac{4}{v} \|x\| \, k_0. \end{aligned}$$

We conclude

$$\|T_n\| = \sup_{\|x\|=1} \|T_n x\| \leq \frac{4}{v} k_0 = c,$$

and c is independent of x. □

Problem 31. Let X be the normed space of all polynomials with norm

$$\|x\| = \max_i |\alpha_i| \,, \text{ where } x(t) = \alpha_k t^k + \cdots + \alpha_0.$$

Show X is not complete.

Solution. Let X be the normed space of all polynomials with norm $\|x\| = \max_i |\alpha_i|$ (the α_i's are the coefficients). Define the linear functional f_n by $f_n(x) = \alpha_0 + \alpha_1 + ... + \alpha_{n-1}$. Then $|f_n(x)| \leq n\|x\|$. Also for each fixed x we have $|f_n(x)| \leq c_x$. On the other hand, for $x(t) = 1 + t + t^2 + ... + t^n$, we have $\|x\| = 1$ and $f_n(x) = n\|x\|$. Hence $\|f_n\| \geq \frac{|f_n(x)|}{\|x\|} = n$. It follows that the sequence $(\|f_n\|)$ is unbounded. The Uniform Boundedness Theorem implies that X is not complete. □

Problem 32. If X, Y are Banach and $(T_n) \subset B(X, Y)$ is a sequence, show TFAE:

1. $(\|T_n\|)$ is bounded.
2. $(\|T_n x\|)$ is bounded $\forall \ x \in X$.
3. $(|g(T_n x)|$ is bounded $\forall \ x \in X \ \forall \ g \in Y'$.

Solution. Recall that in a Banach space X if a sequence (x_n) is such that $(f(x_n))$ is bounded for all $f \in X'$, then $(\|x_n\|)$ is bounded and therefore 3 implies 2. Also, 2 implies 1 by the Uniform Boundedness Theorem. Finally, 1 implies 3 since $|g(T_n x)| \leq \|g\| \|T_n\| \|x\|$. □

4.5 Strong and Weak Convergence

Definition (Strong and Weak Convergence)**.** Let X be a normed space and let $(x_n) \subset X$ be a sequence.

1. (x_n) is **strongly convergent** if $\exists\, x \in X$ s.t.

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

   We denote this property with

$$\lim_{n\to\infty} x_n = x \text{ or } x_n \to x.$$

2. (x_n) is **weakly convergent** if $\exists\, x \in X$ s.t. $\forall\, f \in X'$

$$\lim_{n\to\infty} f(x_n) = f(x).$$

   We denote this property with

$$x_n \xrightarrow{\text{w}} x.$$

Lemma 4.28 (Weak Convergence)**.** *Let $x_n \xrightarrow{w} x$. Then*

1. *The the weak limit x is unique.*
2. *$x_{n_k} \xrightarrow{w} x\ \forall$ subsequences $(x_{n_k}) \subset (x_n)$.*
3. *$(\|x_n\|)$ is bounded.*

Proof.

1. Suppose $x_n \xrightarrow{\text{w}} x$ and $x_n \xrightarrow{\text{w}} y$. Then $f(x_n) \to f(x)$ and $f(x_n) \to f(y) \Rightarrow f(x) = f(y)$. Then $0 = f(x) - f(y) = f(x-y)\ \forall\, f \in X'$. Then by a previous lemma $x = y$.
2. $(f(x_n)) \subset \mathbb{R} \Rightarrow$ all subsequences of $(f(x_n))$ converge and have the same limit.
3. $(f(x_n))$ converges $\Rightarrow |f(x_n)| \leq c_f\ \forall\, n$. Define $g_n \in X''$ by

$$g_n(f) = f(x_n).$$

   Then $|g_n(f)| = |f(x_n)| \leq c_f$, $\Rightarrow (|g_n(f)|)$ is bounded $\forall\, f \in X'$. Then X' complete $\Rightarrow (\|g_n\|)$ bounded by the UBT. Now $\|g_n\| = \|x\|$ and the proof is complete.

□

Theorem 4.29 (Strong and Weak Convergence)**.** *Let X be a normed space and $(x_n) \subset X$.*

1. *$x_n \to x \Rightarrow x_n \xrightarrow{w} x$. The limits are the same.*

2. $x_n \xrightarrow{w} x \not\Rightarrow x_n \to x$ *in general.*

3. $\dim X < \infty \Rightarrow$

$$x_n \xrightarrow{w} x \Rightarrow x_n \to x.$$

Proof.

1. $\|f(x_n) - f(x)\| = \|f(x_n - x)\| \leq \|f\| \|x_n - x\|$. Therefore $x_n \to x \Rightarrow x_n \xrightarrow{\text{w}} x$.

2. We show a counterexample. Let H be a Hilbert space and $(e_n) \subset H$ and orthonormal sequence. Then $\forall\, f \in H', f(x) = \langle x, z\rangle$. In particular, $f(e_n) = \langle e_n, z\rangle$ and by Bessel's inequality

$$\sum_{n=1}^{\infty} |\langle e_n, z\rangle|^2 \leq \|z\|^2 .$$

   Therefore $f(e_n) = \langle e_n, z\rangle \to 0 \;\forall\; f \in H' \Rightarrow e_n \xrightarrow{\text{w}} 0$. But (e_n) does not converge strongly because for $n \neq m$

$$\|e_m - e_n\|^2 = \langle e_m - e_n, e_m - e_n\rangle = 2.$$

3. Suppose $\dim X = k < \infty$ and $x_n \xrightarrow{\text{w}} x$. Let $\{e_1, \ldots, e_k\}$ be a basis for X. Then we can represent

$$\begin{aligned} x_n &= \sum_{i=1}^{k} \alpha_i^{(n)} e_i \\ x &= \sum_{i=1}^{k} \alpha_i e_i. \end{aligned}$$

   By assumption $f(x_n) \to f(x) \;\forall\; f \in X'$. Take the dual basis $\{f_1, \ldots, f_k\}$ defined by

$$f_k(e_i) = \delta_{ki}.$$

   Then

$$\begin{aligned} f_i(x_n) &= \alpha_i^{(n)} \\ f_i(x) &= \alpha_i. \end{aligned}$$

   Hence

$$f_i(x_n) \to f_i(x) \Rightarrow \alpha_i^{(n)} \to \alpha_i.$$

Then

$$\begin{aligned} \|x_n - x\| &= \left\| \sum_{i=1}^{k} (\alpha_i^{(n)} - \alpha_i) e_i \right\| \\ &\le \sum_{i=1}^{k} \left| \alpha_i^{(n)} - \alpha_i \right| \|e_i\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

□

Remark 4.30.

1. *In ℓ^1 strong and weak convergence are equivalent.*

2. *In a Hilbert space H,*

$$x_n \xrightarrow{w} x \text{ iff } \langle x_n, z \rangle \to \langle x, z \rangle \ \ \forall\, z \in H.$$

Lemma 4.31 (Weak Convergence)**.** *Let X be a normed space and let $(x_n) \subset X$. Then $x_n \xrightarrow{w} x$ iff both:*

1. *$(\|x_n\|)$ is bounded.*

2. *$f(x_n) \to f(x)$ $\forall\, f \in M \subset X'$ $\forall\, M$ total .*

Proof. 1. "⇒". Weak convergence implies 1,2 by previous results.

2. "⇐". Suppose 1,2 hold. Let $f \in X'$ be arbitrary and show $f(x_n) \to f(x)$. By 1 we can find $c > 0$ s.t.

$$\begin{aligned} \|x_n\| &\le c \\ \|x\| &\le c. \end{aligned}$$

Since $M \subset X'$ is total, $\forall\, f \in X'$ $\exists\, (f_n) \subset \operatorname{span}(M)$ s.t. $f_n \to f$. Hence $\forall\, \varepsilon > 0$ $\exists\, i$ s.t.

$$\|f_i - f\| < \frac{\varepsilon}{3c}.$$

Moreover, since $f_i \in \operatorname{span}(M)$ $(by 2)$ $\exists\, N$ s.t.

$$|f_i(x_n) - f_i(x)| < \frac{\varepsilon}{3}, \ \forall\, n > N.$$

Then

$$\begin{aligned} |f(x_n)-f(x)| &\leq |f(x_n)-f_i(x_n)|+|f_i(x_n)-f_i(x)|+|f_i(x)-f(x)| \\ &< \|f-f_i\|\,\|x_n\|+\frac{\varepsilon}{3}+\|f_i-f\|\,\|x\| \\ &< \frac{\varepsilon}{3c}c+\frac{\varepsilon}{3}+\frac{\varepsilon}{3c}c \\ &= \varepsilon. \end{aligned}$$

Therefore $x_n \xrightarrow{\text{w}} x$.

□

Definition (Weak Cauchy, Weak Complete)**.**

1. In a normed space X, a sequence $(x_n) \subset X$ is said to be **weak Cauchy** if $\forall\, f \in X'$ the sequence $(f(x_n))$ is Cauchy.
2. A normed space X is said to be **weakly complete** if every weak Cauchy sequence converges weakly in X.

Problem 33. Show X reflexive $\Rightarrow X$ weakly complete.

Solution. Let (x_n) be any weak Cauchy sequence in X. Then $(f(x_n))$ converges for every $f \in X'$. For $x_n \in X$ there is a $g_{x_n} \in X''$ such that $f(x_n) = g_{x_n}(f)$. Hence $(g_{x_n}(f))$ converges, say, $g_{x_n}(f) \to g(f)$. Weak Cauchyness of (x_n) implies the boundedness of (x_n) and then since $\|g_{x_n}\| = \|x_n\|$ we have that g is bounded. Also, g is linear and therefore $g \in X''$. Since X is reflexive, there is an x such that $g(f) = f(x)$. Hence $f(x_n) \to f(x)$. Since $f \in X'$ was arbitrary, this shows that (x_n) converges to x weakly. Since (x_n) was any weak Cauchy sequence, X is weakly complete. □

Definition (Convergence of Operators)**.** Let X, Y be normed spaces, and let $(T_n) \subset B(X, Y)$ be a sequence. We say (T_n) is:

1. **uniformly operator convergent** if (T_n) converges in the norm of $B(X, Y)$.
2. **strongly operator convergent** if $\forall\, x \in X$ the sequence $(T_n x)$ converges strongly in Y.
3. **weakly operator convergent** if $\forall\, x \in X$ the sequence $(T_n x)$ converges weakly in Y.

Remark 4.32. *Uniform $\Rightarrow$strong $\Rightarrow$weak:*

1. $\|T_n - T\| \to 0 \Rightarrow \|(T_n - T)x\| \leq \|T_n - T\|\,\|x\| \to 0.$

2. $\|(T_n - T)x\| \to 0 \Rightarrow \|f(T_n - T)x\| \leq \|f\|\,\|(T_n - T)x\| \to 0$

Example 4.33 (Strong $\not\Rightarrow$ Uniform)**.** Let $X = Y = \ell^2$. Let T_n be "zero-overwrite" operator, i.e., let $(x_i) \in \ell^2$

$$T_n(x_i) = (0, \dots, 0, x_{n+1}, x_{n+2}, \dots).$$

Then $T_n(x_i) \to 0 \ \forall \ (x_i) \in \ell^2$ so (T_n) is strongly operator convergent to the zero operator. However given any n, we can construct $x' \in \ell^2$ s.t. $T_n x = x$, i.e., by making the first n terms of x' zero. Then

$$\|T_n\| = \sup_{x \neq 0} \frac{\|T_n x\|}{\|x\|} \geq 1.$$

So (T_n) does not converge to $0 \in B(X, Y)$.

Example 4.34 (Weak $\not\Rightarrow$ Strong)**.** Let $X = Y = \ell^2$. Let T_n be "zero-shift" operator, i.e., let $(x_i) \in \ell^2$

$$T_n(x_i) = (0, \dots, 0, x_1, x_2, \dots), \text{ n zeros }.$$

Let $(z_i) \in \ell^2$ be another element. Define $f \in \ell^{2\prime}$ by

$$f(x) = \langle x, z \rangle = \sum_{i=1}^{\infty} x_i \overline{z_i}.$$

Then

$$\begin{aligned} f(T_n x) &= \langle T_n x, z \rangle \\ &= \sum_{k=1}^{\infty} x_k \overline{z_{k+n}}. \end{aligned}$$

Compute

$$\begin{aligned} |f(T_n x)|^2 &= |\langle T_n x, z \rangle|^2 \\ &\leq \sum_{k=1}^{\infty} |x_k|^2 \sum_{m=n+1}^{\infty} |z_m|^2 \\ \sum_{m=n+1}^{\infty} |z_m|^2 &\to 0. \end{aligned}$$

Then $f(T_n x) \to 0 = f(0x)$, i.e. (T_n) converges weakly to zero. But consider the sequence $x = (1, 0, 0, \dots)$. Then

$$\|T_n x - T_m x\| = \sqrt{2}, \text{ n} \neq \text{m }.$$

So (T_n) does not converge strongly to zero.

Definition (Strong, Weak and Weak* Convergence of Functionals)**.** Let X be a normed space and let $(f_n) \in X'$ be a sequence.

1. (f_n) is **strongly convergent** to $f \in X'$ if $\|f_n - f\| \to 0$ and we write

$$f_n \to f.$$

2. (f_n) is **weakly convergent** to $f \in X'$ if

$$g(f_n) \to g(f) \ \forall \ g \in X''.$$

3. (f_n) is **weak* convergent** to $f \in X'$ if

$$f_n(x) \to f(x) \ \forall \ x \in X.$$

   We write

$$f_n \xrightarrow{w^*} f.$$

Remark 4.35.

1. *Weak* $\Rightarrow$ *weak*.*
2. *Limit operators. Let* $(T_n) \in B(X,Y)$ *be a sequence.*
   (a) *If* $T_n \to T$ *(uniform) then* $T \in B(X,Y)$.
   (b) *If convergence is strong or weak, it is possible that the limit* T *is unbounded (not continuous) if* X *is not complete.* **This is the why we use Banach spaces**.

Example 4.36. Let X be the subspace of ℓ^2 of sequences with finitely many nonzero terms. Then X is not complete. Define T_n via

$$T_n(x_i) = (x_1, 2x_2, \ldots, nx_n, x_{n+1}, x_{n+2}, \ldots).$$

Then $(T_n) \subset B(X,X)$ converges strong to the unbounded operator T

$$T(x_i) = (ix_i).$$

Lemma 4.37 (Strong Operator Convergence)**.** *Let* X *be a Banach space,* Y *a normed space, and* $(T_n) \subset B(X,Y)$ *a sequence.* (T_n) *strongly operator convergent* $\Rightarrow T \in B(X,Y)$.

Proof. Note

1. T is linear.
2. $T_n x \to Tx \ \forall \ x \in X \Rightarrow T_n x$ is bounded $\ \forall \ x \in X \Rightarrow \|T_n\| \le c$, for some $c \in \mathbb{R}$ by the uniform boundedness theorem.

Now,

$$\begin{aligned}\|T_n x\| &\leq \|T_n\| \|x\| \\ &\leq c \|x\| .\end{aligned}$$

So $\|T_n x\| \leq c \|x\|$. Taking lim on the LHS yields

$$\|Tx\| \leq c \|x\| ,$$

therefore $T \in B(X, Y)$. ☐

Theorem 4.38 (Strong Operator Convergence)**.** *Let X, Y be Banach spaces, and let $(T_n) \subset B(X, Y)$ be a sequence. (T_n) is strongly operator convergent iff*

1. *$(\|T_n\|)$ is bounded.*
2. *$(T_n x)$ is Cauchy $\forall\, x \in M\ \forall\, M$ total in X.*

Proof.

1. $\Rightarrow$. If $T_n x \to Tx\ \forall\, x \in X$ then 1 follows by the UBT and 2 follows trivially.
2. $\Leftarrow$. Suppose $\|T_n\| \leq c\, \forall\, n$. Let $x \in X$ be arbitrary and show $(T_n x)$ converges strongly in Y.

   Let $\varepsilon > 0$ be given. Since $X = \overline{\text{span}\,(M)}$, then $\exists\, y \in \text{span}\,(M)$ s.t.

   $$\|x - y\| < \frac{\varepsilon}{3c}.$$

   Also, $(T_n y)$ Cauchy $\Rightarrow \exists\, N$ s.t.

   $$\|T_n y - T_m y\| < \frac{\varepsilon}{3}.$$

   Then

   $$\begin{aligned}\|T_n x - T_m x\| &\leq \|T_n x - T_n y\| + \|T_n y - T_m y\| + \|T_m y - T_m x\| \\ &< c\frac{\varepsilon}{3c} + \frac{\varepsilon}{3} + c\frac{\varepsilon}{3c} \\ &= \varepsilon.\end{aligned}$$

   Then (T_n) is Cauchy, and since Y is complete, $(T_n x)$ converges in Y.

☐

Corollary 4.39. *Let X be a Banach space and $(f_n) \subset X'$ a sequence which is weak* convergent to f,*

$$f_n \xrightarrow{w^*} f.$$

Then $f \in X'$ iff

1. *$(\|f_n\|)$ is bounded.*
2. *$(f_n x)$ is Cauchy $\forall\, x \in M\ \forall\, M$ total in X.*

4.6 The Open Mapping Theorem

Definition (Open Mapping)**.** Let X, Y be metric spaces, $T : \mathcal{D}(T) \to Y$, $\mathcal{D}(T) \subset X$. Then T is said to be an **open mapping** if $B \subset \mathcal{D}(T)$ open $\Rightarrow T(B) \subset Y$ open.

Theorem 4.40 (Open mapping)**.** *Let X, Y be Banach spaces and $T \in B(X, Y)$ s.t. $T : X \overset{onto}{\longrightarrow} Y$. Then T is an open mapping. Hence, if T is injective then T is bijective and so T^{-1} is continuous and so $T^{-1} \in B(X, Y)$.*

Lemma 4.41 (Open Unit Ball)**.** *Let X, Y be Banach spaces and $T \in B(X, Y)$ s.t. $T : X \overset{onto}{\longrightarrow} Y$ and $B_0 = B(0, 1) \subset X$. Then $T(B_0) \subset Y$ is open and $0 \in T(B_0)$.*

Proof. Let $A \subset X$. Define

1. $\alpha A = \{x \in X | x = \alpha a, a \in A\}$
2. $A + w = \{x \in X | x = a + w, a \in A\}$.

Consider $B_1 = B(0, \frac{1}{2} \subset X$, and let $x \in X$ be arbitrary. Then choose k s.t. $x \in kB_1$, e.g., $k > 2\|x\|$. Then

$$X = \bigcup_{k=1}^{\infty} kB_1.$$

Since T is surjective and linear,

$$\begin{aligned} Y &= T(X) \\ &= T\left(\bigcup_{k=1}^{\infty} kB_1\right) \\ &= \bigcup_{k=1}^{\infty} kT(B_1) \\ &= \bigcup_{k=1}^{\infty} \overline{kT(B_1)}. \end{aligned}$$

Then by the category theory, Y complete $\Rightarrow Y$ not meager. Then at least one of $\overline{kT(B_1)}$ contains an open ball $\Rightarrow T(B_1)$ contains an open ball, say

$$B_* = B(y_0, \varepsilon) \subset \overline{T(B_1)}.$$

Then

$$B_* - y_0 = B(0, \varepsilon) \subset \overline{T(B_1)} - y_0.$$

With these sets constructed, we want to show

$$\overline{T(B_1)} - y_0 \subset \overline{T(B_0)}.$$

Let $y \in \overline{T(B_1)} - Y_0$. Then $y + y_0 \in \overline{T(B_1)}$. □

Problem 34. Let X, Y be Banach spaces and $T \in B(X,Y)$ injective. Show that $T^{-1} : \mathcal{R}(T) \to X$ is bounded iff $\mathcal{R}(T)$ is closed in Y.

Solution.

1. If $R(T)$ is closed in Y, it is complete, and boundedness follows from the Open Mapping Theorem.

2. Assume T^{-1} to be bounded, $y \in \overline{R(T)} \subset Y$, (y_n) in $R(T)$ such that $y_n \to y$, and $x_n = T^{-1}y_n$. Since T^{-1} is continuous and X is complete, (x_n) converges, say, $x_n \to x$. Since T is continuous, $y_n = Tx_n \to Tx$. Hence $y = Tx \in R(T)$, so that $R(T)$ is closed because $y \in \overline{R(T)}$ was arbitrary.

□

4.7 Closed Graph Theorem

Definition (Closed Linear Operator)**.** Let X, Y be normed spaces and $T : \mathcal{D}(T) \to Y$ a linear operator with $\mathcal{D}(T) \subset X$. T is said to be **closed** if the graph of T, denoted ΓT

$$\Gamma T = \{(x,y) | x \in \mathcal{D}(T), y = Tx\}$$

is closed in $X \times Y$.

Theorem 4.42 (Closed Graph Theorem)**.** *Let X, Y be Banach spaces and let $T : \mathcal{D}(T) \to Y$ be a closed linear operator. If $\mathcal{D}(T)$ is closed in X then T is bounded.*

Proof. Note that $X \times Y$ is normed space with norm

$$\|(x,y)\| = \|x\| + \|y\| \tag{11}$$

First we show $X \times Y$ is complete with norm (11). Let $z_n = (x_n, y_n)$ and $(z_n) \in X \times Y$ Cauchy. Let $\varepsilon > 0$ be given. $\exists\, N$ s.t.

$$\|z_n - z_m\| = \|x_n - x_m\| + \|y_n - y_m\| < \varepsilon \;\forall\; n, m > N.$$

Then $(x_n), (y_n)$ are Cauchy in X, Y respectively. Since X, Y are Banach, these sequences converge, say

$$x_n \to x \in X, y_n \to y \in Y.$$

Then setting $z = (x, y)$,

$$z_n \to z \in X \times Y.$$

Since (z_n) was arbitrary, $X \times Y$ is complete.

Now, by assumption ΓT is closed in $X \times Y$ and $\mathcal{D}(T)$ is closed in X. Therefor $\Gamma T, \mathcal{D}(T)$ are complete. Consider $P : \Gamma T \to \mathcal{D}(T)$ given by the map

$$(x, Tx) \mapsto x.$$

Then

1. P is linear. Obvious.

2. P is bounded.

$$\begin{aligned} \|P(x,Tx)\| &= \|x\| \\ &\leq \|x\| + \|Tx\| \\ &= \|(x,Tx)\|. \end{aligned}$$

3. P is bijective with inverse $P^{-1} : \mathcal{D}(T) \to \Gamma T$ given by the map

$$x \mapsto (x, Tx).$$

Since ΓT and $\mathcal{D}(T)$ are complete, P^{-1} is bounded, say

$$\|(x,Tx)\| \leq b\|x\|.$$

Then

$$\begin{aligned} \|Tx\| &\leq \|Tx\| + \|x\| \\ &= \|(x,Tx)\| \\ &\leq b\|x\|, \end{aligned}$$

$\forall\ x \in \mathcal{D}(T)$.

Then T is bounded, as required. ☐

Theorem 4.43 (Closed Linear Operator)**.** *Let X, Y be normed spaces, $T : \mathcal{D}(T) \to Y$ a linear operator, and $\mathcal{D}(T) \subset X$. Then T is closed iff*

1. *Given $(x_n) \subset \mathcal{D}(T)$.*

2. *If $x_n \to x$ and $Tx_n \to y$ then*

$$x \in \mathcal{D}(T) \text{ and } Tx = y.$$

Proof. $z \in \overline{\Gamma T}$ iff $\exists$ sequence $(z_n) \subset \Gamma T$

$$z_n = (x_n, Tx_n)$$

s.t.

$$z_n \to z.$$

Hence, $x_n \to x$ and $Tx_n \to y$. Then $z = (x,y) \in \Gamma T$ iff

$$x \in \mathcal{D}(T) \text{ and } y = Tx.$$

☐

Example 4.44 (Differential Operator)**.** Let $X = C[0,1]$ and let $T : \mathcal{D}(T) \to X$ be given by the map

$$x \mapsto x'.$$

Note $\mathcal{D}(T)$ is the space of continuously differentiable functions. Then

1. T is bounded. Obvious.

2. T is closed. Suppose $x_n \to x$ and $Tx_n = x_n' \to y$. Then

$$\begin{aligned}
\int_0^1 y(\tau)d\tau &= \int_0^1 \lim_{n\to\infty} x_n'(\tau)d\tau \\
&= \lim_{n\to\infty} \int_0^1 x_n'(\tau)d\tau \\
&= x(t) - x(0).
\end{aligned}$$

$\Rightarrow$

$$x(t) = x(0) + \int_0^1 y(\tau)d\tau.$$

$\Rightarrow$

$$\begin{aligned}
x &\in \mathcal{D}(T) \\
x' &= y
\end{aligned}$$

Remark 4.45. *Boundedness does not imply closedness. To see this, let $T : \mathcal{D}(T) \to \mathcal{D}(T) \subset X$ be the identity operator where $\mathcal{D}(T)$ is proper, dense subset of X. Then T is linear and bounded.*

However T is not closed. Take $x \in X - \mathcal{D}(T)$ and let $(x_n) \subset \mathcal{D}(T)$ be s.t.

$$x_n \to x.$$

Then

$$Tx_n = x_n \to x \notin \mathcal{D}(T).$$

Lemma 4.46 (Closed Operator)**.** *Let X, Y be normed spaces, let $T \in B(\mathcal{D}(T), Y)$ with $\mathcal{D}(T) \subset X$.*

1. If $\mathcal{D}(T)$ is a closed subset of X then T is close.

2. If T is closed and Y is complete then $\mathcal{D}(T)$ is a closed subset of X.

Proof.

1. If $(x_n) \subset \mathcal{D}(T)$ and $x_n \to x$ and (Tx_n) converges then

(a) $x \in \overline{\mathcal{D}(T)} = \mathcal{D}(T)$ since $\mathcal{D}(T)$ is closed.

(b) $Tx_n \to Tx$ since T is closed.

$\Rightarrow T$ is closed.

2. For $x \in \overline{\mathcal{D}(T)}$ $\exists$ $(x_n) \subset \mathcal{D}(T)$ s.t.

$$x_n \to x.$$

Since T is bounded,
$$\|Tx_n - Tx\| \leq \|T\| \|x_n - x_m\|.$$
Therefore (Tx_n) is Cauchy, and since Y is complete

$$Tx_n \to y \in Y.$$

Since T is closed we have $x \in \mathcal{D}(T)$ and $Tx = y$. Hence $\mathcal{D}(T)$ is closed because $y \in \overline{\mathcal{D}(T)}$ was arbitrary.

□

Problem 35. Let X and Y be normed spaces and let $T : X \to Y$ be a closed, linear operator. Show the following.

1. The image B of a compact subset $C \subset X$ is closed in Y.

2. The inverse image A of compact subset $K \subset Y$ is closed in X.

Solution. 1. Consider any $a \in \bar{A}$. Let $a_n \to a$, where $a_n \in A$. Let $c_n \in C$ be such that $a_n = Tc_n$. Since C is compact, (c_n) has a subsequence (c_{n_k}) which converges, say, $c_{n_k} \to c \in C$. Also $Tc_{n_k} \to a$, and $Tc = a \in A$ because T is closed by assumption.

2. Consider any $b \in \bar{B}$. Let $b_n \to b$, where $b_n \in B$. Let $k_n = Tb_n$. Since K is compact, (k_n) has a subsequence (k_{n_i}) which converges, say, $k_{n_i} \to k \in K$. Also $b_{n_i} \to b$, and $Tb = k \in K = T(B)$ by the closedness of T, so that $b \in B$ and B is closed.

□

5 Exam 1

Problem 1 (4 Points)**.** *Let $M \subset l^\infty$ be the subspace consisting of all sequences $x = (\xi_i)$ with at most finitely many nonzero terms. Is M complete?*

Solution. Let $M \subset l^\infty$, and $x = (1, 1/2, 1/3, ...) = (\xi_i)$. Consider the sequence (x_n), where $x_n = (1, 1/2, ..., 1/n, 0, ...)$. Then $x_n \in M$ and $d(x_n, x) = 1/(n+1)$, i.e., we have that $x_n \to x$, but $x \notin M$. It follows that M is not closed. □

Problem 2 (4 Points)**.** *Does*

$$d(x,y) = \int_a^b |x(t) - y(t)| dt$$

define a metric or pseudometric on X if X is

1. *the set of all real-valued continuous functions on $[a, b]$*

2. *the set of all real-valued Riemann integrable functions on $[a, b]$?*

Solution. d is a metric on $C[a, b]$, because if the integral is 0, then $|x(t) - y(t)| = 0$ for all $t \in [a, b]$. On the other hand d is a pseudo-metric on $R[a, b]$, for example if $x(a) = 1$ and $x(t) = 0$ everywhere else on $[a, b]$, then the integral of $|x(t)|$ is 0. □

Problem 3 (4 Points)**.** *If X is a compact metric space and $M \subset X$ is closed, show that M is compact.*

Solution. Let $(x_n) \subset M \subset X$. Since X is compact (x_n) has a convergent subsequence, say $(x_{n_k}) \subset M$, and $x_n \to x$. M is closed, so we must have $x \in M$. It follows that M is compact. □

Problem 4 (4 Points)**.** *Let $T : X \to Y$ be a linear operator and $dimX = dimY = n < \infty$. Show that $R(T) = Y$ iff T^{-1} exists.*

Solution.

1. Suppose that T^{-1} exists and let $\{e_1, e_2, ..., e_n\}$ a basis for X. We show that $\{Te_1, Te_2, ..., Te_n\}$ are linearly independent (and therefore $R(T) = Y$).

   Suppose that $\alpha_1 Te_1 + ... + \alpha_n Te_n = T(\alpha_1 e_1 + ... + \alpha_n e_n) = 0$. T is bijective by assumption, so we have that $\alpha_1 e_1 + ... + \alpha_n e_n = 0$, and then $\alpha_1 = ... = \alpha_n = 0$. It follows that $\{Te_1, ..., Te_n\}$ are linearly independent.

2. Suppose that $R(T) = Y$, and $(y_1, y_2, ..., y_n)$ is a basis for Y. Then there exist $x_1, x_2, ..., x_n \in X$ such that $Tx_i = y_i$, for $i = 1, 2, ..., n$.

We can show that $\{x_1, x_2, ..., x_n\}$ are linearly independent using the same argument as above. It follows that $\{x_1, x_2, ..., x_n\}$ is a basis for X.

Suppose that $Tx = 0$. Then $Tx = T(\alpha_1 x_1 + ... + \alpha_n x_n) = \alpha_1 y_1 + ... + \alpha_n y_n = 0$, and $\alpha_1 = ... = \alpha_n = 0$. It follows that $x = 0$ and T is injective. Also, T is surjective by assumption. It follows that T^{-1} exists.

□

Problem 5 (4 Points)**.** *Show that the operator $T : l^\infty \to l^\infty$ defined by*

$$y = (\eta_i) = Tx, \; \eta_i = \frac{\xi_i}{i}, \; x(\xi_i),$$

is linear and bounded What can you say about T^{-1}?.

Solution. Let $T : l^\infty \to l^\infty$, $x = (\xi_i) \in l^\infty$, $Tx = (\xi_i/i)$. Then T is bounded, linear, injective, but not surjective, e.g., $y = (1, 1, 1, ..) \in l^\infty$ is not in $R(T)$. T^{-1} is only defined on $R(T) \subset l^\infty$. T^{-1} is not bounded, because for $x_n = (\xi_{ni}$, where $\xi_{ni} = 1$ if $i = n$ and 0 otherwise we get

$$\frac{\|t^{-1} x_n\|}{\|x_n\|} = n.$$

□

6 Exam 2

Problem 1 (4 Points)**.** *If x and y are different vectors in a finite dimensional vector space X, show that there is a linear functional f on X such that $f(x) \neq f(y)$.*

Solution. Let $x, y \in X$, $x \neq y$, $dim X = n$.

1. Suppose that $x \neq \alpha y$, for any α. Then x and y are linearly independent and we can construct a basis $\{x = e_1, y = e_2, e_3, ..., e_n\}$, and a corresponding dual basis $\{f_1, f_2, ..., f_n\}$, where $f_i(e_j) = \delta_{ij}$. Then $f_1(x) = 1 \neq f_1(y)$.

2. If $x = \alpha y$, $\alpha \neq 1$, then we take $\{x = e_1, e_2, e_3, ..., e_n\}$, and $\{f_1, f_2, ..., f_n\}$. Then $f_1(x) = 1 \neq f_1(y) = \alpha$.

□

Problem 2 (4 Points)**.** *Let X and Y be normed spaces and $T_n : X \to Y$, $n = 1, 2, 3, ..,$ be bounded linear operators. Show that convergence $T_n \to T$ implies that for every $\epsilon > 0$ there is an N such that for all $n > N$ and all x in any given closed ball we have $\|T_n x - Tx\| < \epsilon$.*

Solution. Let B be a closed ball in X. B is bounded, i.e., if $x \in B$, then $\|x\| < K$ for some K. Let N be such that $\|T_n - T\| < \frac{\epsilon}{K}$ for all $n > N$. Then

$$\|T_n x - Tx\| = \|(T_n - T)x\| \leq \|T_n - T\|\|x\| < \epsilon.$$

□

Problem 3 (4 Points)**.** *Show that $y \perp x_n$ and $x_n \to x$ together imply $x \perp y$.*

Solution. Suppose that $y \perp x_n$ and $x_n \to x$. Then

$$\langle x, y\rangle = \lim_{n\to\infty} \langle x_n, y\rangle = 0.$$

□

Problem 4 (4 Points)**.** *Let M be a total set in an inner product space X. If $\langle v, x\rangle = \langle w, x\rangle$ for all $x \in M$, show that $v = w$.*

Solution. We have by assumption that $\langle v - w, x\rangle = 0$ for all $x \in M$. M is total in X, so $v - w \in \overline{span} M = X$. By the continuity of the inner product we have $\langle v - w, v - w\rangle = \|v - w\|^2 = 0$. □

Problem 5 (4 Points)**.** *If S and T are bounded self-adjoint operators on a Hilbert space H and α and β are real, show that $L = \alpha S + \beta T$ is self-adjoint.*

Solution. Let $L = \alpha S + \beta T$. Then we have

$$L^* = (\alpha S + \beta T)^* = (\alpha S)^* + (\beta T)^* = \bar{\alpha} S^* + \bar{\beta} T^* = \alpha S^* + \beta T^* = \alpha S + \beta T = L.$$

□

7 Final Exam

Problem 1 (4 Points)**.** *What can you say about the reflexivity of l^p, $1 \le p \le \infty$?*

Solution. l^p, For $1 < p < \infty$ is reflexive using the fact that the dual space of l^p is l^q, where $\frac{1}{p} + \frac{1}{q} = 1$. On the other hand, l^1 and l^∞ are nonreflexive spaces. For l^1 we can use the argument that a separable normed space with a nonseparable dual cannot be reflexive. Concerning l^∞, we know that c_0, the space of convergent sequences with limit equal to zero is a subspace of l^∞ and that the dual of c_0 is l^1. It follows that the dual of l^∞ is larger than l^1, and therefore l^∞ is not reflexive. □

Problem 2 (4 Points)**.** *Let X be a separable Banach space and $M \subset X'$ a bounded set. Show that every sequence of elements of M contains a subsequence which is weak* convergent to an element of X'.*

Solution. Any sequence (f_n) in M is bounded, say$\|f_n\| \le r$. Since X is separable, it contains a countable dense subset V, which we can arrange in a sequence (x_m). Since

$$|f_n(x_m)| \le \|f_n\| \|x_m\| \le r\|x_m\|,$$

we see that for fixed m the sequence $(f_n(x_m))$ is bounded, so that it has a subsequence A_1 which converges at x_1, and A_1 has a subsequence A_2 which converges at x_2, ...etc.; hence $(f_{n_k}(x))$, where $f_{n_1} \in A_1, f_{n_2} \in A_2, ...$, is a subsequence which converges at every element of V. Since V is dense in X and X is complete the statement follows. □

Problem 3 (4 Points)**.** *Show that an open mapping need not map closed sets onto closed sets.*

Solution. The mapping $T : R^2 \to R$ defined by $(x_1, x_2) \to (x_1)$ is open, it maps the closed set $\{(x_1, x_2) | x_1 x_2 = 1\} \subset R^2$ onto the set $R - \{0\}$ which is not closed in R. □

Problem 4 (4 Points)**.** *Let X and Y be normed spaces and X compact. If $T : X \to Y$ is a bijective closed linear operator, show that T^{-1} is bounded.*

Solution. T^{-1} is closed, being the inverse of a closed operator. Hence $T^{-1} : Y \to X$, where T^{-1} is closed and X is compact, is bounded. □

Problem 5 (4 Points)**.** *Let f be an integrable function on the measure space $(X, \mathcal{B}, \mu)$. Show that given $\epsilon > 0$, there is a $\delta > 0$ such that for each measurable set E with $\mu E < \delta$ we have*

$$|\int_E f| < \epsilon.$$

Solution. Let f be an integrable function on the measure space (X, B, μ). Let $f_n(x) = f(x)$ if $|f(x)| \leq n$, $f_n(x) = n$ if $f(x) \geq n$, $f_n(x) = -n$ if $f(x) \leq in$. Then $|f - f_n| \to 0$ a.e. and $|f - f_n| \leq |f|$ for all n. By the dominated convergence theorem we have

$$\int |f - f_n| \to o.$$

Let $\epsilon > 0$ be given. We can pick N such that

$$\int |f - f_n| < \frac{\epsilon}{2} \text{ for all } N.$$

Let $\delta < \frac{\epsilon}{2N}$. If E is measurable with $\mu E < \delta$, then

$$|\int_E f| = |\int_E (f_N + (f - f_N))| \leq \int_E |f_N| + \int_E |f - f_N| \leq \delta N + \int |f - f_N| \leq \epsilon.$$

□