GAUSSIAN QUADRATURE (1)

- Objective: Approximate an integral as a sum

$$\int_a^b f(x)\,w(x)\,dx \approx \sum_{i=0}^{m-1} w_i\,f(x_i)$$

as accurately as possible, given m points

- ▷ w is a positive-valued weight function
- ▷ Tools:
  - ∘ Polynomial interpolation d'après Lagrange
  - ∘ Orthogonal polynomials
- ▷ Values of the weights w_i will be derived that are optimal in the sense of maximizing the degree of a polynomial for which the above integration formula is exact

GAUSSIAN QUADRATURE (2)

- It is always possible to find weights that make the formula

$$\int_a^b f(x)w(x)dx = \sum_{i=0}^{m-1} w_i f(x_i)$$

exact if f is a polynomial of degree $\le m-1$ by solving the linear system

$$\int_a^b w(x)\,dx = \sum_{i=0}^{m-1} w_i$$

$$\int_a^b x\,w(x)\,dx = \sum_{i=0}^{m-1} w_i\,x_i$$

$$\vdots$$

$$\int_a^b x^{m-1}\,w(x)\,dx = \sum_{i=0}^{m-1} w_i\,x_i^{m-1}$$

LAGRANGIAN INTERPOLATION (1)

- Interpolation of sampled values of a function f:
  - ▷ Given sampled values $f(x_i)$ at sampling points $x_0, \ldots, x_{m-1}$
  - ▷ Define a function e_k that interpolates the values $0, \ldots, 0, 1, 0, \ldots, 0$ (1 in location k, 0 elsewhere):

$$e_k(x_i) = \delta_{ki} = \begin{cases} 1, \text{if } x_i = x_k; \\ 0, \text{if } x_i \neq x_k. \end{cases}$$

  - ▷ The polynomial of lowest degree that meets the requirements on e_k is

$$e_k(x) = \frac{\displaystyle\prod_{i=0}^{k-1} (x - x_i) \prod_{j=k+1}^{m-1} (x - x_j)}{\displaystyle\prod_{r=0}^{k-1} (x_k - x_r) \prod_{s=k+1}^{m-1} (x_k - x_s)}$$

LAGRANGIAN INTERPOLATION (2)

- Interpolation of sampled values of a function f:
  - ▷ Given sampled values $f(x_i)$ at sampling points $x_0, \ldots, x_{m-1}$
  - ▷ The interpolating polynomial of lowest degree is

$$F(x) = \sum_{k=0}^{m-1} f(x_k)\, e_k(x)$$

where

$$e_k(x_i) = \begin{cases} 1, \text{if } x_i = x_k; \\ 0, \text{if } x_i \neq x_k. \end{cases}$$

  - ▷ The **remainder** is

$$R(x) = f(x) - F(x) = \frac{f^{(m)}(\alpha)}{m!} S(x) \quad \text{for} \quad x \in \{x_0, \ldots, x_{m-1}\}$$

where $\alpha \in [x_0, x_{m-1}]$ and the **sampling polynomial** is

$$S(x) = \prod_{i=0}^{m-1} (x - x_i)$$

GAUSSIAN QUADRATURE (3)

- Method: Integrate the Lagrange interpolation formula
  - ▷ Integrate $f(x) = F(x) + R(x)$:

$$\int_a^b f(x)\,w(x)\,dx = \int_a^b F(x)\,w(x)\,dx + \int_a^b R(x)\,w(x)\,dx$$
$$= \sum_{i=0}^{m-1} w_i\,f(x_i) + \frac{f^{(m)}(\alpha)}{m!}\int_a^b S(x)\,w(x)\,dx$$

  - ▷ This formula is exact $(S(x) = 0)$ if f is a polynomial of degree $\leq m - 1$
- Since degree$[e_k] = m - 1$,

$$\int_a^b e_k(x)\,w(x)\,dx = \sum_{i=0}^{m-1} w_i\,e_k(x_i) = \sum_{i=0}^{m-1} w_i\,\delta_{ki} = w_k$$

GAUSSIAN QUADRATURE (4)

- To derive the optimal values of the sample points x_i:
  - ▷ Suppose temporarily that f is a polynomial of degree $\leq 2m-1$
  - ▷ Divide f by p_m: $$f(x) = q(x)\,p_m(x) + r(x)$$
    The quotient q and remainder r are polynomials of degree $\leq m-1$
  - ▷ Since p_m is orthogonal to all polynomials of degree $\leq m-1$,

$$\int_a^b q(x)\,p_m(x)\,w(x)\,dx = 0$$

$$\begin{aligned}\int_a^b f(x)\,w(x)\,dx &= \int_a^b q(x)\,p_m(x)\,w(x)\,dx + \int_a^b r(x)\,w(x)\,dx \\ &= \int_a^b r(x)\,w(x)\,dx \\ &= \sum_{i=0}^{m-1} w_i\,r(x_i) \\ &= \sum_{i=0}^{m-1} w_i[f(x_i) - q(x_i)\,p_m(x_i)]\end{aligned}$$

GAUSSIAN QUADRATURE (5)

- The optimal values of the sample points x_i are the zeros of p_m, because

$$\int_a^b f(x)\,w(x)\,dx = \sum_{i=0}^{m-1} w_i\big[f(x_i) - q(x_i)\,p_m(x_i)\big]$$

  ▷ The choice

$$\boxed{x_i = i^{\text{th}} \text{ zero of } p_m}$$

  makes the formula

$$\boxed{\int_a^b f(x)\,w(x)\,dx \approx \sum_{i=0}^{m-1} w_i\, f(x_i)}$$

  exact if f is a polynomial of degree $\leq 2m-1$