DERIVATION OF THE PARAXIAL WAVE EQUATION (1)

- Begin with Maxwell's equations:

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= 4\pi\rho_{\mathrm{free}} - 4\pi\nabla \cdot \mathbf{P} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \\
\nabla \times \mathbf{H} &= \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \\
&= \frac{1}{c}\frac{\partial}{\partial t}\left[(1 + 4\pi\chi_L)\mathbf{E} + 4\pi\mathbf{P}_{NL}\right] \\
&= \frac{1}{c}\frac{\partial}{\partial t}\left[n^2\mathbf{E} + 4\pi\mathbf{P}_{NL}\right]
\end{aligned}$$

- When expanding $\nabla \times (\nabla \times \mathbf{E})$, neglect $\nabla(\nabla \cdot \mathbf{E})$ **(paraxial approximation)**

$$\Rightarrow \left[\nabla^2 - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{E} = \frac{4\pi}{c^2}\frac{\partial^2 \mathbf{P}_{NL}}{\partial t^2}$$

- Problem: the paraxial approximation neglects $\nabla(\nabla \cdot \mathbf{P}_{NL})$, which may not be negligible if the nonlinearity is large.

DERIVATION OF THE PARAXIAL WAVE EQUATION (2)

- Introduce slowly varying envelope functions:

$$\mathbf{E} = \mathrm{Re}\left[\hat{\mathbf{e}}\mathcal{E}(\mathbf{r}_T,\, z,\, t)e^{i(kz-\omega t)}\right] = \mathrm{Re}\left[\hat{\mathbf{e}}\mathcal{E}e^{-i\omega t'}\right]$$

$$\mathbf{P}_{NL} = \mathrm{Re}\left[-i\hat{\mathbf{e}}\mathcal{P}(\mathbf{r}_T,\, z,\, t)e^{i(kz-\omega t)}\right]$$

$$= \mathrm{Re}\left[\hat{\mathbf{e}}(C - iS)e^{-i\omega t' + i\phi}\right]$$

  where $k = n\omega/c$,

$$t' := t - \frac{nz}{c} = \text{retarded time}, \quad \text{and} \quad \mathcal{E} = |\mathcal{E}|e^{i\phi}.$$

  It follows that

$$\mathcal{P} = i(C - iS)e^{i\phi}$$

- Assume that the envelopes $\mathcal{E}$ and $\mathcal{P}$ vary slowly on the scale of an optical wavelength or an optical period **(approximation of slowly varying amplitude and phase)**:

$$\left|\frac{\partial \mathcal{E}}{\partial z}\right| \ll k|\mathcal{E}|; \quad \left|\frac{\partial \mathcal{E}}{\partial t}\right| \ll \omega|\mathcal{E}|$$

- Result:

$$\left[\nabla_T^2 + 2ik\left(\frac{\partial}{\partial z} + \frac{n}{c}\frac{\partial}{\partial t}\right)\right]\mathcal{E} = \frac{4\pi i k^2}{n^2}\mathcal{P}$$

DERIVATION OF THE PARAXIAL WAVE EQUATION (3)

- Introduce retarded time (time measured relative to the arrival time at z of a light pulse unaffected by the nonlinear polarization $\mathbf{P}_{NL}$) :

$$\begin{aligned} t' &:= t - \frac{nz}{c} \\ z' &:= z \end{aligned}$$

- Functions transform as follows:

$$\begin{aligned} \bar{f}(\mathbf{r}_T,\, z',\, t') &:= f(\mathbf{r}_T,\, z,\, t) \\ \frac{\partial \bar{f}}{\partial z'} &= \frac{\partial f}{\partial z} + \frac{n}{c}\frac{\partial f}{\partial t} \end{aligned}$$

- Resulting **paraxial** or **one-way wave equation**:

$$\left[\nabla_T^2 + 2ik\frac{\partial}{\partial z'}\right]\overline{\mathcal{E}} = \frac{4\pi i k^2}{n^2}\overline{\mathcal{P}}$$

  where $\overline{\mathcal{E}}$ and $\overline{\mathcal{P}}$ are functions of $\mathbf{r}_T$, z' and t'.

- Susceptibility (if applicable):

$$\chi = \frac{\mathcal{P}}{i\mathcal{E}}$$

DERIVATION OF THE PARAXIAL WAVE EQUATION (4)

- Interpretation of the "disappearance" of $\partial/\partial t'$ from the paraxial wave equation

$$\left[\nabla_T^2 + 2ik\frac{\partial}{\partial z'}\right]\mathcal{E} = \frac{4\pi i k^2}{n^2}\mathcal{P}$$

  - ▷ Both the polarization $\mathcal{P}$ and the field $\mathcal{E}$ still depend on the retarded time. A steady-state approximation has **not** been made in going to a comoving frame.
  - ▷ A steady-state or other dynamical approximation may be applied in the calculation of $\mathcal{P}$.
  - ▷ If $\mathcal{P}$ is calculated by a method that allows for coherence or hysteresis effects, then different retarded times are coupled through $\mathcal{P}$.

DERIVATION OF THE PARAXIAL WAVE EQUATION (5)

- Equivalence of the homogeneous paraxial wave equation (HPWE)

$$\frac{\partial \mathcal{E}}{\partial z} = \frac{i}{2k} \nabla_T^2 \mathcal{E}$$

to the Fresnel limit of the Kirchhoff diffraction integral:

- The formal solution of the HPWE is

$$\mathcal{E}(\mathbf{r}_T,\, z) = \exp\left\{\frac{iz}{2k}\nabla_T^2\right\} \mathcal{E}(\mathbf{r}_T,\, 0)$$

- In terms of the Fourier transform

$$\mathcal{E}(\mathbf{r}_T,\, 0) = (2\pi)^{-1} \int e^{i\mathbf{q}\cdot\mathbf{r}_T}\, \tilde{\mathcal{E}}(\mathbf{q},\, 0)\, d^2q,$$

the formal solution becomes

$$\mathcal{E}(\mathbf{r}_T,\, z) = (2\pi)^{-1} \int \exp\left\{i\left[-\frac{q^2 z}{2k} + \mathbf{q}\cdot\mathbf{r}_T\right]\right\} \tilde{\mathcal{E}}(\mathbf{q},\, 0)\, d^2q$$

$$= \text{Fourier transform of a product}$$

- Convolution theorem:

$$(2\pi)^{-1} \int e^{i\mathbf{q}\cdot\mathbf{r}_T}\, \tilde{f}(\mathbf{q})\tilde{g}(\mathbf{q})\, d^2q = (2\pi)^{-1} \int f(\mathbf{r}_T - \mathbf{r}'_T) g(\mathbf{r}'_T)\, d^2r'_T$$

where

$$\tilde{f}(\mathbf{q}) := (2\pi)^{-1} \int e^{-i\mathbf{q}\cdot\mathbf{r}_T}\, f(\mathbf{r}_T)\, d^2r_T$$

DERIVATION OF THE PARAXIAL WAVE EQUATION (6)

- The formal solution of the HPWE, the Fourier transform

  $$(2\pi)^{-1} \int \exp\left\{i\left[-\frac{\mathbf{q}^2 z}{2k} + \mathbf{q}\cdot\mathbf{r}_T\right]\right\} d^2q = -i\frac{k}{z}\exp\left[i\frac{k\mathbf{r}_T^2}{2z}\right]$$

  and the convolution theorem imply that

  $$\mathcal{E}(\mathbf{r}_T,\, z) = -i\frac{k}{2\pi z}\int \exp\left\{i\frac{k}{2z}(\mathbf{r}_T - \mathbf{r}'_T)^2\right\}\mathcal{E}(\mathbf{r}'_T,\, 0)\, d^2r'_T$$

- The Fresnel-Kirchhoff diffraction integral, evaluated in the forward direction, is

  $$\mathcal{E}(\mathbf{r}_T,\, z) = -i\frac{k}{2\pi}\int \frac{e^{iks}}{s}\mathcal{E}(\mathbf{r}'_T,\, 0)\, d^2r'_T$$

  where

  $$s^2 = (\mathbf{r}_T - \mathbf{r}'_T)^2 + z^2$$

- In the Fresnel limit

  $$s \approx z + \frac{(\mathbf{r}_T - \mathbf{r}'_T)^2}{2z}$$

  the Fresnel-Kirchhoff diffraction integral reduces to the integral form of the solution of the HPWE