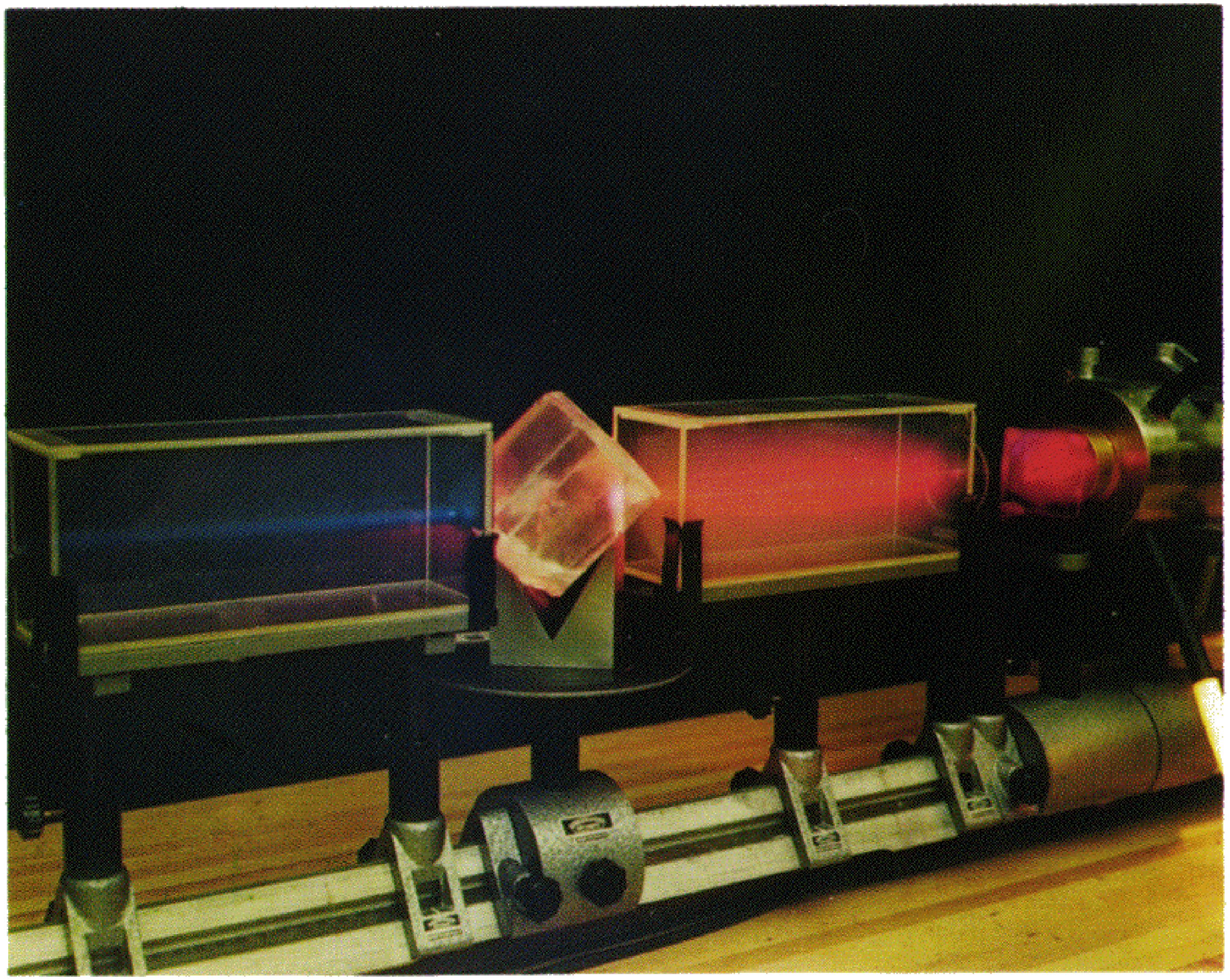
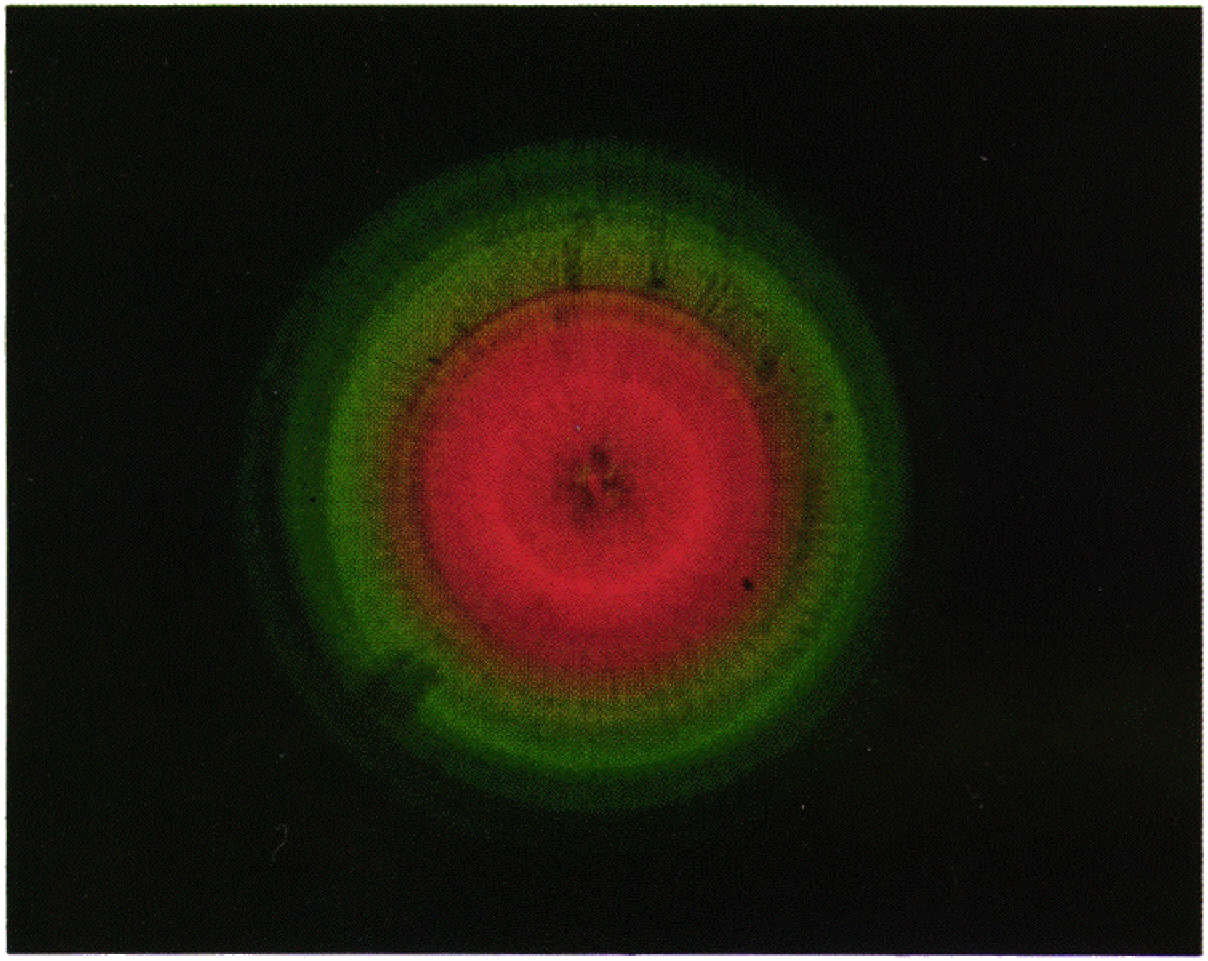
NONLINEAR OPTICS

- Comprises some of the most striking effects in physics
  - ▷ Light of a single color enters a transparent substance
  - ▷ Different colors emerge on the other side
- Examples:
  - ▷ Second harmonic generation: ω in, 2ω out
  - ▷ Stimulated Raman scattering: ω_L in, $\omega_S = \omega_L - \omega_v$ out
    - ◦ ω_L is called the "pump" because energy is pumped from ω_L to ω_S
    - ◦ ω_S is called the Raman-Stokes frequency for historical reasons
      - ⋄ Initially the Raman-Stokes beam experiences exponential gain
    - ◦ ω_v is the frequency of a molecular vibration or an optical phonon
    - ◦ At finite temperatures there's also a small amount of "anti-Stokes" light at the frequency $\omega_A = \omega_L + \omega_v$
  - ▷ Stimulated Brillouin scattering: ω_L in, $\omega_B = \omega_L - \omega_a$ out
    - ◦ ω_a is the frequency of a hypersonic (high-frequency acoustic) phonon (11 GHz in silica fiber)

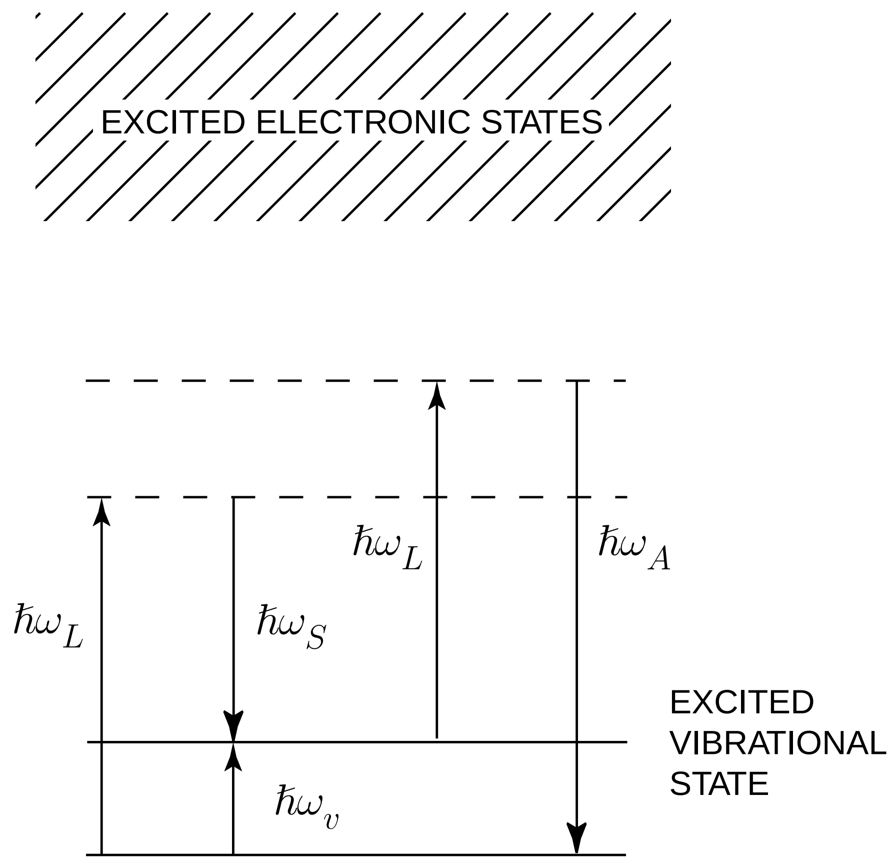


Laser beam enters a crystal of ammonium dihydrogen phosphate as red light and emerges as blue—the second harmonic. Courtesy of R. W. Terhune.



Light created when a ruby laser is focused in a cell of benzene emerges to form a pattern of brightly colored rings. New frequencies come from Raman resonances in the benzene. Courtesy of R. W. Terhune.

STIMULATED RAMAN SCATTERING (2)



NONLINEAR EFFECTS IN OPTICAL FIBERS

- Light interacts strongly with light when

$$\text{gain coefficient} \cdot \frac{P}{A} \cdot L \gg 1$$

- Nonlinear gain coefficient $\leq 4 \times 10^{-9}\,\text{cm/W}$ in silica
- Nonlinear effects can be large in a single-mode fiber:
  - ▷ Low loss ⇒ long propagation distances:

$$L \geq 20\,\text{km}\,(2 \times 10^{6}\,\text{cm})$$

  - ▷ Small core area:

$$A < 10^{-6}\,\text{cm}^2$$

  - ▷ Optical amplification ⇒ high intensity:

$$P/A > 10^{4}\,\text{W/cm}^2$$

MEASUREMENT OF OPTICAL NONLINEARITIES IN FIBER

- Basic approach for measuring small-signal nonlinear gain:
  - ▷ Launch a pump beam into the fiber
  - ▷ Probe the gain with a weak pulse at the frequency (and in the direction) of maximum gain
- Example of experimental setup is shown on next slide
- Example of eye-diagram degradation due to SBS is shown on the slide after the next one
- Reference: Daniel A. Fishman and Jonathan A. Nagel, *Journal of Lightwave Technology* **11**, 1721–1728 (1993)

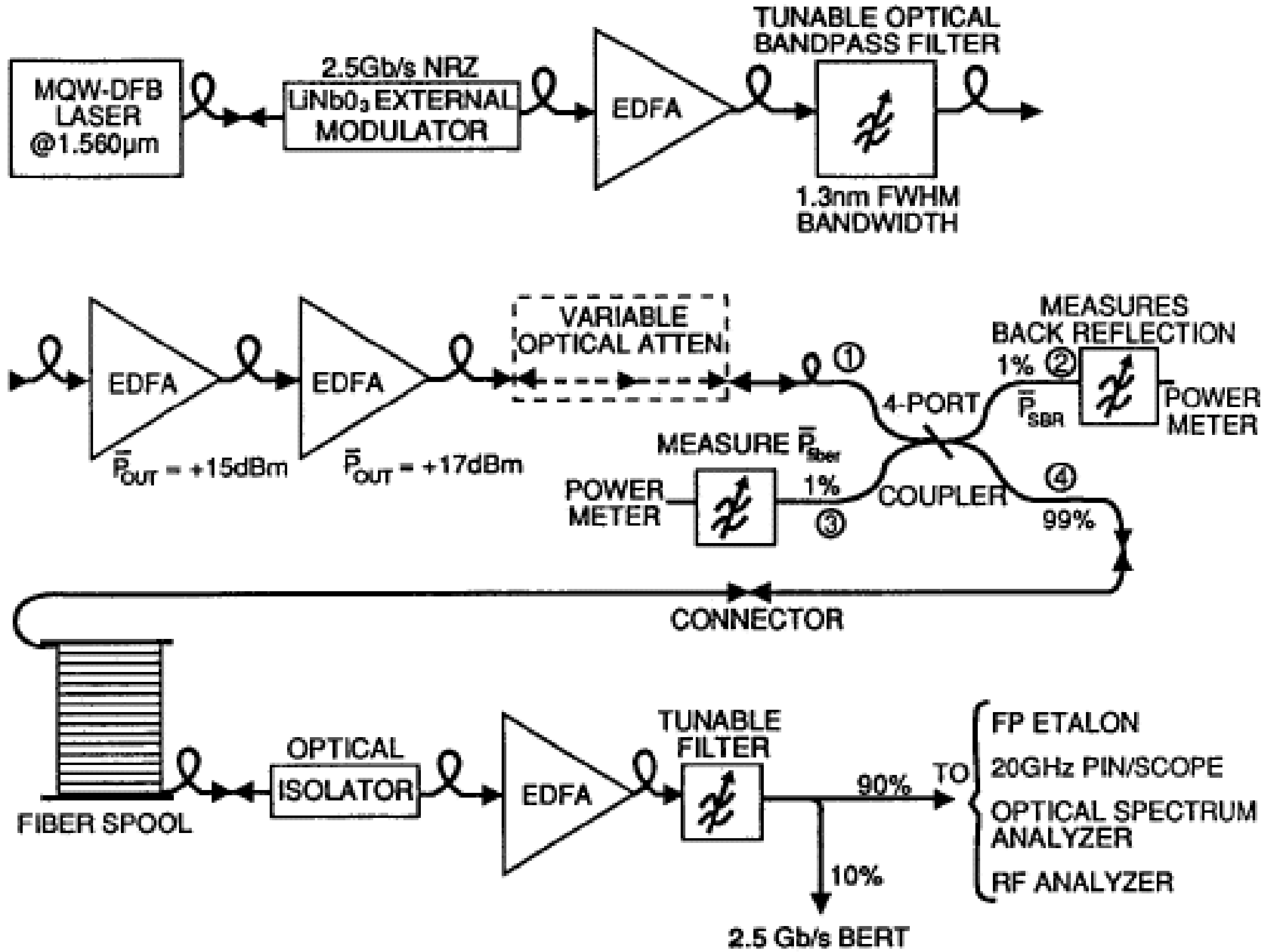


Fig. 1. Schematic of experimental setup. All the EDFAs have optical isolators on input and output ports to eliminate amplification of the backward-propagating signals. The optical terminations into the power meters are isolated to reduce residual reflections into port 2. The arrows and --×-- markings indicate demountable optical connections, comprised of ST and Biconic connectors. The launch and reflected power reading are calibrated to take into account the connector losses, which are measured with the fiber spool removed. In addition to the MQW-DFB CW laser source, an external-cavity laser (ECL) with a 420-kHz (FWHM) linewidth was also used.

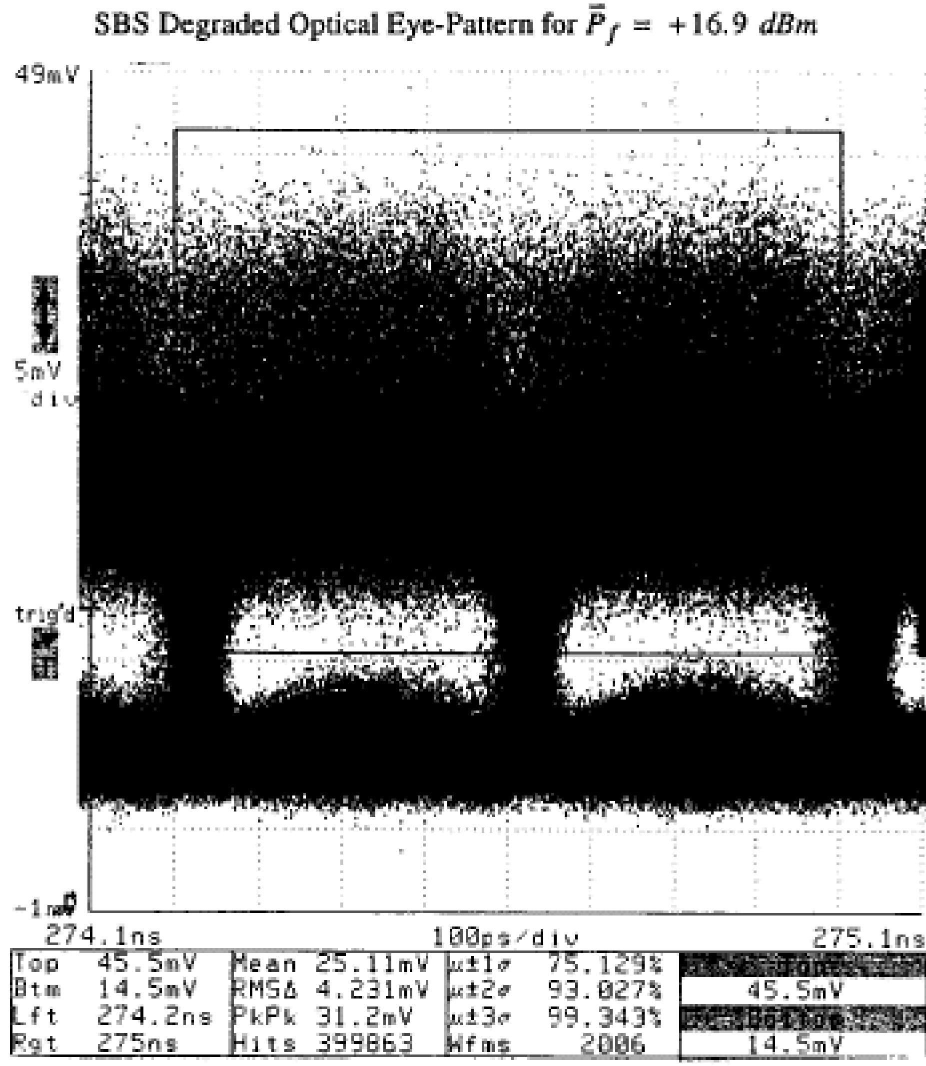


Fig. 4. Typical degraded eye pattern due to SBS for $\bar{P}_f = 16.9$ dB. The signal-to-noise ratio is obtained by measuring the average "on" and "off" voltage levels and then dividing the difference by the sum of the rms fluctuations. In this case the SNR is 3.2 (10.2 dB) which corresponds to a BER floor at 10^{-5}.

ELECTROMAGNETIC UNITS (1)

- **MKS units**:
  - ▷ Common in engineering and physics
  - ▷ Useful when calculating electric & magnetic field strengths
  - ▷ Introduce the impedance of the propagation medium,

$$Z_0 = \sqrt{\frac{\mu}{\epsilon}},$$

    into many propagation equations
  - ▷ Have no advantage in optical calculations

ELECTROMAGNETIC UNITS (2)

- **Gaussian cgs units**:
  - ▷ Hard to use correctly when calculating field strengths
  - ▷ OK for optical calculations
  - ▷ Used in (almost) all Bell Labs papers on nonlinear optics in fibers
- Conversion factors:
  - ▷ 1 Coulomb (C) $= 3 \times 10^9$ stat-Coulombs (stC)
  - ▷ 1 volt (V) $= (299.793)^{-1}$ statvolts (sV)
  - ▷ 1 ampere (A) $= 0.1$ abampere (abA)
  - ▷ Charge on electron: $e = 1.6 \times 10^{-19}$ C$= 4.8 \times 10^{-10}$ stC

MAXWELL'S EQUATIONS (1)

- Maxwell's equations involve $\mathbf{E}$ (electric field), $\mathbf{D}$ (electric displacement), $\mathbf{H}$ (magnetic field) and $\mathbf{B}$ (magnetic induction):

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday's Law)}$$

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{J} \quad \text{(Maxwell's version of Ampère's Law)}$$

$$\nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{D} = 0 \quad \text{(no magnetic monopoles or free charges)}$$

- Note the use of cgs units (as usual in nonlinear optics)

MAXWELL'S EQUATIONS (2)

- In most transparent materials,

$$\mathbf{B} = \mathbf{H}, \quad \mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$$

- Use $\mathbf{J} = \sigma\mathbf{E}$ to describe distributed losses of all kinds
- The electric polarization $\mathbf{P}$ is the sum of linear and nonlinear terms:

$$\mathbf{P} = \mathbf{P}_L + \mathbf{P}_{NL}$$

- The linear electric polarization is

$$\mathbf{P}_L = \begin{cases} \chi^{(1)}\mathbf{E} & \text{(instantaneous response)} \\ \int_{-\infty}^{\infty} \tilde{\chi}^{(1)}(\omega)\,\tilde{\mathbf{E}}(\omega)e^{-i\omega t}d\omega & \text{(frequency-dependent response)} \end{cases}$$

where $\chi^{(1)}$ is the first-order (linear) electric susceptibility

MAXWELL'S EQUATIONS (3)

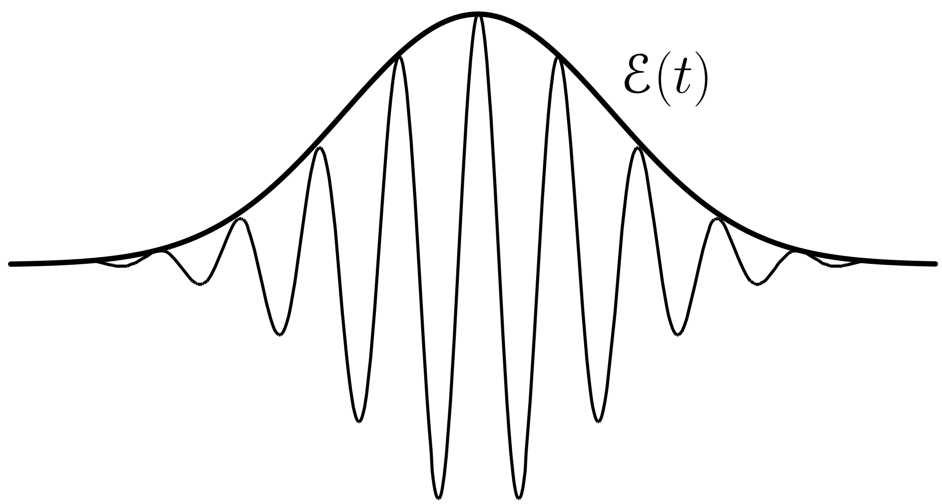
- Vector wave equation:

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \frac{4\pi}{c}\frac{\partial \mathbf{J}}{\partial t} - \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}(\mathbf{P}_L + \mathbf{P}_{NL})$$

- In the paraxial approximation,

$$\underbrace{\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}}_{\text{wavepropagation}} = \underbrace{\frac{4\pi}{c}\frac{\partial \mathbf{J}}{\partial t} + \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{P}_L}_{\text{linear}} + \underbrace{\frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{P}_{NL}}_{\text{nonlinear}}$$

SLOWLY-VARYING-ENVELOPE APPROXIMATION (SVEA)



- $\mathcal{E}$ is the envelope: $\mathcal{E}(t) = A(t)e^{i\phi(t)}$, where A and ϕ are real
- **SVEA**: Neglect $|\partial\mathcal{E}/\partial t|$ in comparison with $|\omega\mathcal{E}|$
  - ▷ This is equivalent to assuming that $\mathcal{E}$ changes slowly on a time scale of $2\pi/\omega$
  - ▷ If this is a good approximation, then the field is **quasimonochromatic**

COMPLEX SUSCEPTIBILITY (1)

- Quasimonochromatic electric field E and induced dipole moment per unit volume, P ("polarization"):

$$E = \mathrm{Re}\,[\mathcal{E}e^{-i\omega t}], \quad P_L = \mathrm{Re}\,[-i\mathcal{P}_L e^{-i\omega t}]$$

- Electric polarization and electric flux density in Gaussian cgs units:

$$P = \chi E, \qquad D = E + 4\pi P$$

- Linear electric susceptibility:

$$\chi^{(1)} = \frac{\mathcal{P}_L}{i\mathcal{E}} \quad \text{where } \mathcal{E} \text{ is constant in } t$$

- Relative electric permittivity (dielectric "constant"):

$$\epsilon = 1 + 4\pi\chi^{(1)} = 1 + 4\pi\left[\chi^{(1)\prime} + i\chi^{(1)\prime\prime}\right]$$

COMPLEX SUSCEPTIBILITY (2)

- Wave equation for a quasimonochromatic field:

$$\left(\nabla^2 - \frac{\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right) E = 0; \text{ then } \left(\nabla^2 + \frac{\epsilon\omega^2}{c^2}\right) \mathcal{E} = 0$$

- Plane-wave solution:

$$\mathcal{E} = \mathcal{E}_0 \exp\left[i\frac{\omega}{c}\sqrt{\epsilon}\, z\right]$$

  ▷ Real part of $\sqrt{\epsilon}$ gives **dispersion**:

$$n = \mathrm{Re}\,[\sqrt{\epsilon}\,] \approx 1 + 2\pi\chi^{(1)\prime}$$

  ▷ Imaginary part of $\sqrt{\epsilon}$ gives **absorption** (if $\chi^{(1)\prime\prime} > 0$) or **gain** (if $\chi^{(1)\prime\prime} < 0$):

$$\alpha = 2\omega\,\mathrm{Im}\,[\sqrt{\epsilon}\,]/c \approx \frac{2\pi\omega}{nc}\chi^{(1)\prime\prime}$$

ELECTRIC POLARIZATION (1)

- The j^{th} vector component of the linear electric polarization $\mathbf{P}_L$:

$$P_{L,j}(\mathbf{E}) = \sum_k \chi^{(1)}_{jk} E_k$$

  - ▷ $E_k = k^{\text{th}}$ vector component of the electric field $\mathbf{E}$
  - ▷ in isotropic media (glasses, gases), $\chi^{(1)}_{11} = \chi^{(1)}_{22} = \chi^{(1)}_{33}$ are the only non-zero components
  - ▷ Birefringence occurs when the components of $\chi^{(1)}$ are unequal (crystalline or strained media)

- The j^{th} vector component of the nonlinear electric polarization $\mathbf{P}_{NL}$:

$$P_{NL,j}(\mathbf{E}) = 2\sum_{k,l} \chi^{(2)}_{jkl} E_k E_l + 4\sum_{k,l,m} \chi^{(3)}_{jklm} E_k E_l E_m + \cdots$$

  - ▷ The factors of 2 for $\chi^{(2)}$ and 4 for $\chi^{(3)}$ are conventional

ELECTRIC POLARIZATION (2)

- Second-order nonlinear electric polarization:

$$P^{(2)}_{NL,j}(\mathbf{E}) = 2\sum_{k,l} \chi^{(2)}_{jkl} E_k E_l$$

▷ In a medium with inversion symmetry (no preferred direction),

$$\mathbf{P}_{NL}(-\mathbf{E}) = -\mathbf{P}_{NL}(\mathbf{E})$$

▷ But

$$P^{(2)}_{NL,j}(-\mathbf{E}) = 2\sum_{k,l} \chi^{(2)}_{jkl}(-E_k)(-E_l)$$
$$= +P^{(2)}_{NL,j}(\mathbf{E})$$

▷ Therefore, in unstrained glasses there are no second-order nonlinear-optical effects:

$$\boxed{\chi^{(2)}_{jkl} = 0}$$

SELECTED SECOND-ORDER PROCESSES

- Second-order nonlinear polarization:

$$P^{(2)}_{NL,n} = 2 \sum_{k,l} \chi^{(2)}_{nkl}(-\omega_n, \omega_k, \omega_l) E_k E_l$$

- Frequency convention:

$$-\omega_n + \omega_k + \omega_l = 0$$

- Selected $\chi^{(2)}$ processes:

$-\omega_n, \omega_k, \omega_l$	identification	susceptiblity
$-\omega, 0, \omega$	Pockels	$\chi^{(2)\prime}$
$0, \omega, -\omega$	OR	$\chi^{(2)\prime}$
$-2\omega, \omega, \omega$	SHG	$\chi^{(2)\prime}$
$-\omega_3, \omega_1, \omega_2$	SFG	$\chi^{(2)\prime}$
$-\omega_3, \omega_1, -\omega_2$	DFG, PG	$\chi^{(2)\prime}$

Pockels –Pockels (linear electro-optic) effect
OR –optical rectification
SHG –second harmonic generation
SFG –sum-frequency generation
DFG –difference-frequency generation
PG –parametric gain

ELECTRIC POLARIZATION (3)

- Third-order nonlinear electric polarization:

$$P^{(3)}_{NL,j}(\mathbf{E}) = 4\sum_{k,l,m} \chi^{(3)}_{jklm} E_k E_l E_m$$

- The **third-order nonlinear susceptibility** $\chi^{(3)}$ is complex:

$$\chi^{(3)} = \chi^{(3)\prime} - i\chi^{(3)\prime\prime}$$

  ▷ $\chi^{(3)\prime}$: **Nonlinear dispersion**: Self-phase modulation, cross-phase modulation, four-wave mixing

    ∘ Value: $4\chi^{(3)\prime}_{1111} \approx 6 \times 10^{-15}$ (Gaussian units, esu)

  ▷ $\chi^{(3)\prime\prime}$: **Nonlinear gain/loss**: Raman-Stokes gain, anti-Stokes absorption

    ∘ Value: $4\chi^{(3)\prime\prime} \approx 5.5\times10^{-16}$ (maximum) at a shift of 440 cm^{-1} (Gaussian units, esu)

SELECTED THIRD-ORDER PROCESSES

- Third-order nonlinear polarization:

$$P_{NL,n}^{(3)} = 4 \sum_{k,l,m} \chi_{nklm}^{(3)}(-\omega_n,\, \omega_k,\, \omega_l,\, \omega_m) E_k E_l E_m$$

- Frequency convention:

$$-\omega_n + \omega_k + \omega_l + \omega_m = 0$$

- Selected $\chi^{(3)}$ processes:

$-\omega_n,\, \omega_k,\, \omega_l,\, \omega_m$	identification	susceptiblity
$-\omega,\, \omega,\, -\omega,\, \omega$	SPM	$\chi^{(3)\prime}$
$-\omega_2,\, \omega_1,\, -\omega_1,\, \omega_2$	XPM	$\chi^{(3)\prime}$
$-\omega_B,\, \omega_L,\, -\omega_L,\, \omega_B$	SBS, SRS	$\chi^{(3)\prime\prime}$
$-\omega_a,\, \omega_p,\, \omega_p,\, -\omega_s$	CARS	$\chi^{(3)\prime\prime}$
$-\omega_a,\, \omega_p,\, \omega_p,\, -\omega_s$	PG	$\chi^{(3)\prime}$
$-\omega_4,\, \omega_1,\, \omega_2,\, -\omega_3$	FWM	$\chi^{(3)\prime}$

SPM –self-phase modulation
XPM –cross-phase modulation
SBS –stimulated Brillouin scattering
SRS –stimulated Raman scattering
CARS –coherent anti-Stokes Raman scattering
PG –parametric gain
FWM –four-wave mixing

ELECTRIC POLARIZATION (4)

- $\chi^{(3)}$ is frequency-dependent:

$$\chi^{(3)}_{nklm} = \chi^{(3)}_{nklm}(-\omega_n,\, \omega_k,\, \omega_l,\, \omega_m)$$

  ▷ $\chi^{(3)}$ with no subscripts or arguments means $\chi^{(3)\prime}_{xxxx}(-\omega_n,\, \omega_k,\, \omega_l,\, \omega_m)$

  ▷ In SiO_2, $\chi^{(3)\prime}$ is the sum of a large frequency-independent term and a smaller frequency-dependent term

  ▷ $\chi^{(3)\prime\prime}(-\omega_S, \omega_L, -\omega_L, \omega_S)$ has a broad peak near $\omega_L - \omega_S = 440\ \text{cm}^{-1}$

ELECTRIC POLARIZATION (5)

- Third-order nonlinear electric polarization:

$$P^{(3)}_{NL,j}(\mathbf{E}) = 4 \sum_{k,l,m} \chi^{(3)}_{jklm} E_k E_l E_m$$

- In isotropic media (glasses, gases),

$$\begin{aligned} \chi^{(3)}_{xxxx} &= 3\chi^{(3)}_{xxyy} = 3\chi^{(3)}_{xyyx} = 3\chi^{(3)}_{xyxy} \\ &= 3\chi^{(3)}_{xxzz} = 3\chi^{(3)}_{xzzx} = 3\chi^{(3)}_{xzxz} \end{aligned}$$

are the only non-zero components that contribute to $P^{(3)}_{NL,x}$

Then

$$\begin{aligned} P^{(3)}_{NL,x} &= 4\chi^{(3)}_{xxxx}(E_x)^3 + 4 \sum_{j=y,z} (\chi^{(3)}_{xxjj} + \chi^{(3)}_{xjjx} + \chi^{(3)}_{xjxj})(E_j)^2 E_x \\ &= 4\chi^{(3)}_{xxxx} \left[(E_x)^2 + (E_y)^2 + (E_z)^2\right] E_x \\ &= 4\chi^{(3)}_{xxxx} \left(\mathbf{E} \cdot \mathbf{E}\right) E_x \end{aligned}$$

NONLINEAR GAIN (1)

- Poynting's theorem:

$$\nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E} - \frac{1}{4\pi}\left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}\right)$$

$$\mathbf{S} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{H})$$

$I = S_z$ is the intensity (power/unit area) incident on a surface perpendicular to the z axis

- Power/unit area of a wave propagating along the z axis:

$$\frac{\partial I}{\partial z} = -\sigma \mathbf{E}^2 - \frac{\partial}{\partial t} U_{EM} - \mathbf{E} \cdot \frac{\partial \mathbf{P}_{NL}}{\partial t}$$

- Electromagnetic energy density:

$$U_{EM} = \frac{1}{8\pi}(\epsilon_L \mathbf{E}^2 + \mathbf{H}^2)$$

NONLINEAR GAIN (2)

- If we average

$$\frac{\partial I}{\partial z} = -\sigma \mathbf{E}^2 - \frac{\partial}{\partial t} U_{EM} - \mathbf{E} \cdot \frac{\partial \mathbf{P}_{NL}}{\partial t}$$

over a few optical cycles, $\partial U_{EM} / \partial t$ drops out for all but femtosecond pulses. Then

$$\boxed{\frac{\partial I}{\partial z} = \left\langle -\sigma \mathbf{E}^2 - \mathbf{E} \cdot \frac{\partial \mathbf{P}_{NL}}{\partial t} \right\rangle}$$

describes the propagation of the intensity, I (power/unit area), for all but the shortest laser pulses

NONLINEAR GAIN (3)

- Assume a quasimonochromatic, plane-polarized field and electric polarization:

$$\mathbf{E} = \hat{\mathbf{x}}\,\mathrm{Re}\,[\mathcal{E}e^{-i\omega t}], \quad \mathbf{P}_{NL} = \hat{\mathbf{x}}\,\mathrm{Re}\,[-i\mathcal{P}e^{-i\omega t}]$$

- Insert into the equation for $\partial I/\partial z$ and average over a few cycles:

$$\boxed{\frac{\partial\langle I\rangle}{\partial z} = -\frac{\sigma}{2}|\mathcal{E}|^2 - \frac{\omega}{4}\left(\mathcal{E}^*\mathcal{P} + \mathcal{E}\mathcal{P}^*\right)}$$

- Intensity in terms of slowly varying field envelope:

$$\langle I\rangle = \frac{c\sqrt{\epsilon}}{8\pi}|\mathcal{E}|^2$$

NONLINEAR GAIN (4)

- For the nonlinear effects that matter in telecommunications,

$$-i\mathcal{P} \approx (\chi^{(3)\prime} + i\chi^{(3)\prime\prime})|\mathcal{F}|^2\mathcal{E}$$

where $\mathcal{F}$ is an optical-frequency electric field and $\chi^{(3)}$ is an effective third-order nonlinear susceptibility.

- Use $|\mathcal{E}|^2 = 8\pi\langle I\rangle/(c\sqrt{\epsilon}\,)$:

$$\frac{\partial\langle I\rangle}{\partial z} = -\underbrace{\frac{4\pi\sigma}{c\sqrt{\epsilon}}\langle I\rangle}_{\text{linear loss}} - \underbrace{\frac{4\pi\omega}{c\sqrt{\epsilon}}\chi^{(3)\prime\prime}|\mathcal{F}|^2\langle I\rangle}_{\text{nonlinear gain or loss}}$$

Nonlinear gain occurs when $\chi^{(3)\prime\prime} < 0$

Nonlinear loss occurs when $\chi^{(3)\prime\prime} > 0$

NONLINEAR GAIN (5)

- Nonlinear conversion depends on intensity = power per unit area:

$$I = P/A$$

- Units of nonlinear gain:

$$[g] = \frac{[\text{area}]}{[\text{power}][\text{length}]} = \text{cm/W}$$

- Nonlinear processes with gain include
  - ▷ Stimulated scattering
    - ○ stimulated Brillouin scattering(SBS)
    - ○ stimulated Raman scattering (SRS)
  - ▷ Parametric processes
    - ○ four-wave mixing (FWM)

NONLINEAR GAIN (6)

- Change of power/unit area of a probe beam after propagating through a distance Δz:

$$\Delta I_1 = +g\, I_2\, I_1\, \Delta z$$

$$g = \text{nonlinear gain coefficient}$$

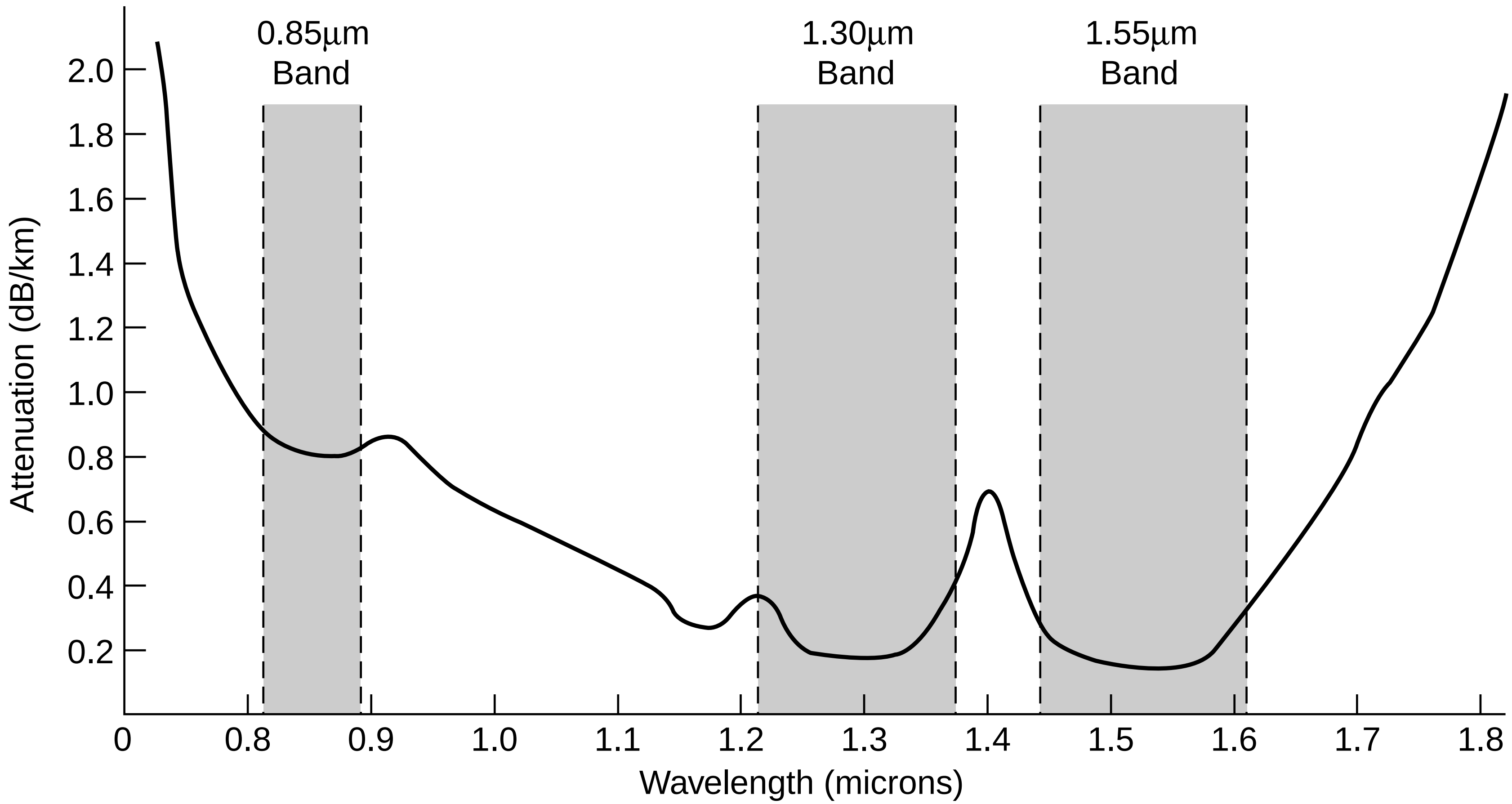
$$I_1 = \text{probe intensity} = P_1/A_e$$

$$I_2 = \text{pump intensity} = P_2/A_e$$

$$A_e = \text{effective area of beam}$$

ATTENUATION MECHANISMS (1)

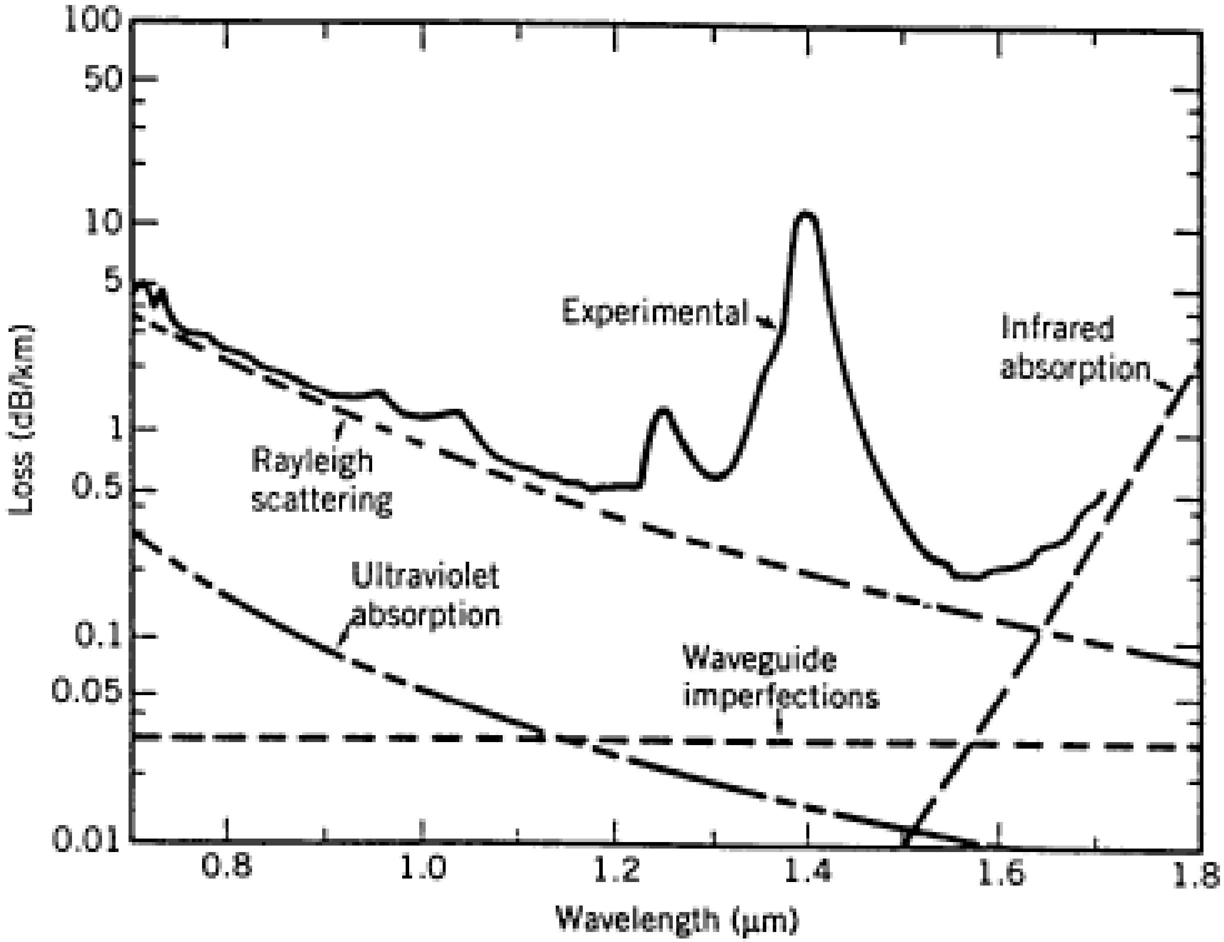
- Material absorption
  - ▷ Intrinsic
  - ▷ Extrinsic
- Rayleigh scattering
- Waveguide scattering
  - ▷ Mie scattering
  - ▷ Microbending
  - ▷ Macrobending



Attenuation of light in silica fiber in the near-infrared region

After: *Computer Networks*, 3rd ed. by Andrew S. Tanenbaum, © 1996 Prentice Hall

ATTENUATION MECHANISMS (2)



EFFECTS OF PUMP ATTENUATION (1)

- Nonlinear conversion depends on intensity = power per unit area:

$$\frac{dI_1}{dz} = \underbrace{g}_{\text{gain}} \underbrace{I_2(0)e^{-\alpha z}}_{\text{pump}} I_1 \quad \underbrace{-\alpha I_1}_{\text{attenuation}}$$

I_1 = probe intensity = P_1/A_e $\quad$ I_2 = pump intensity = P_2/A_e

α = attenuation coefficient $\quad$ A_e = effective area of wave

- Differential equation for $J_1(z) = e^{\alpha z} I_1(z)$:

$$\frac{dJ_1}{dz} = g I_2(0) e^{-\alpha z} J_1$$

▷ Integrate from $z = 0$ to $z = L$:

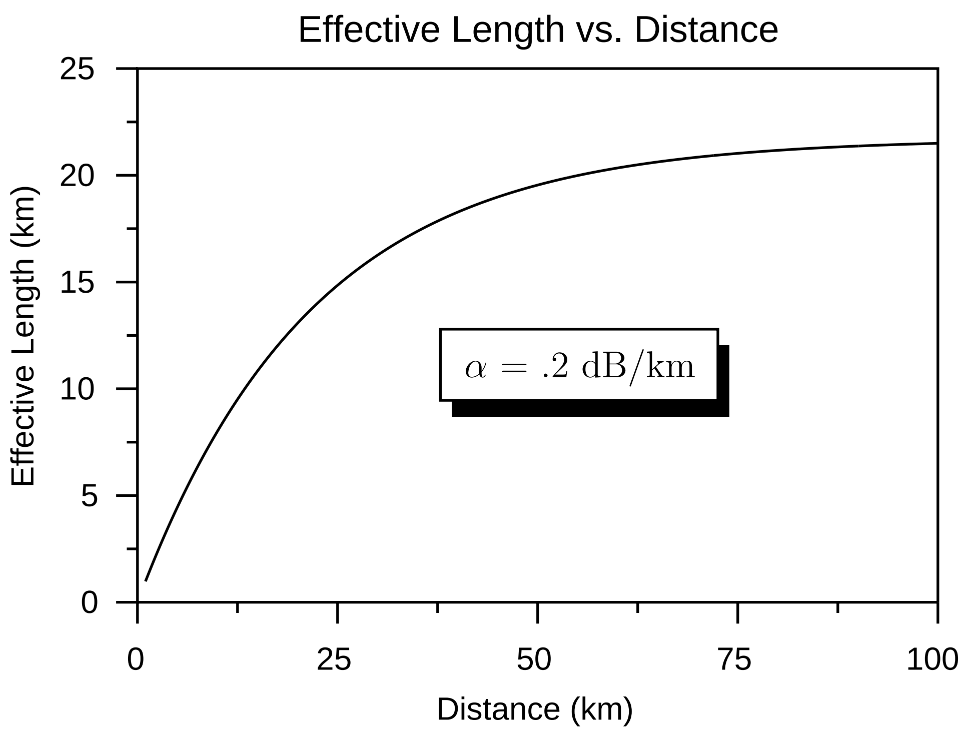
$$I_1(L) = I_1(0)\, e^{-\alpha L} \exp\left[g \int_0^L I_2(0) e^{-\alpha z}\, dz \right] = I_1(0)\, e^{g I_2(0) L_e(L) - \alpha L}$$

$$\text{where} \quad L_e(L) = \frac{1 - e^{-\alpha L}}{\alpha} = \textbf{effective length}$$

EFFECTIVE LENGTH vs. DISTANCE

- Effective length of a fiber of physical length L:

$$L_e(L) = \frac{1 - e^{-\alpha L}}{\alpha}$$



EFFECTS OF PUMP ATTENUATION (2)

- Nonlinearly-generated power:

$$P_1(L) = P_1(0)\, e^{gP_2(0)L_e(L)/A_e - \alpha L}$$

- Effective length: $L_e(L) = \alpha^{-1}(1 - e^{-\alpha L})$
  - ▷ Short propagation distance ($L \ll \alpha^{-1}$):

$$L_e(L) \approx L$$

  - ∘ P_1 grows exponentially if gain > loss, *i.e.*, if

$$gP_2(0)/A_e > \alpha$$

  - ▷ Long propagation distance ($L \gg \alpha^{-1}$):

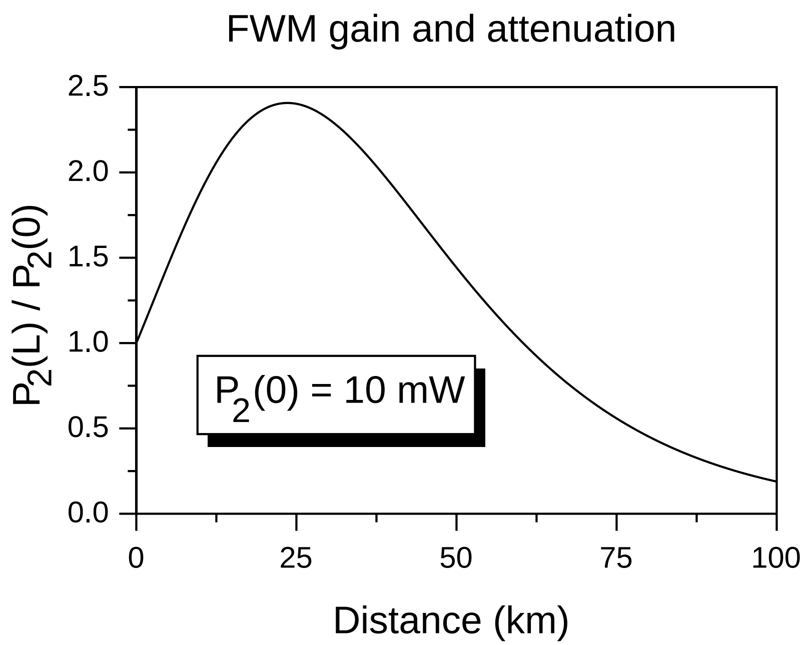
$$L_e(L) \approx \alpha^{-1}$$

  - ∘ Linear attenuation: $P_1(L) = \left(P_1(0)\, e^{gP_2(0)/(\alpha A_e)}\right) e^{-\alpha L}$

EFFECTS OF PUMP ATTENUATION (3)

- Nonlinearly-generated power:

$$P_1(L) = P_1(0)\, e^{gP_2(0)L_e(L)/A_e - \alpha L}$$



CONVERSION OF ATTENUATION UNITS

- Conversion from cgs to practical units:

$$e^{-\alpha_{\mathrm{cm}^{-1}} L_{\mathrm{cm}}} = 10^{-(0.1\alpha_{\mathrm{dB/km}})L_{\mathrm{km}}}$$

$$\Rightarrow \alpha_{\mathrm{cm}^{-1}} L_{\mathrm{cm}} = (\log_e 10)(0.1\alpha_{\mathrm{dB/km}})L_{\mathrm{km}}$$

$$\boxed{\alpha_{\mathrm{cm}^{-1}} = 2.303 \times 10^{-6} \alpha_{\mathrm{dB/km}}}$$

NONLINEAR DISPERSION (1)

- When $\mathbf{E}$ is quasimonochromatic and attenuation is weak,

$$\mathrm{Re}\left[\sqrt{\epsilon}\right] \approx n_0 + \tfrac{1}{2} n_2 |\mathcal{E}|^2 \approx \sqrt{\epsilon^{(1)\prime} + 8\pi\chi^{(3)\prime}|\mathcal{E}|^2}$$

$$n_2 = \textbf{nonlinear refractive index} \approx \frac{8\pi\chi^{(3)\prime}}{n_0} \quad \text{where} \quad n_0 = \sqrt{\epsilon^{(1)\prime}}$$

  - ▷ Propagation of a plane wave over a short distance:

$$\begin{aligned}\mathcal{E}(\mathbf{r}_T, z, t) &= \mathcal{E}(\mathbf{r}_T, 0, t) \exp\left[i\frac{\omega}{c}\sqrt{\epsilon(\mathbf{r}_T, t)}z\right] \\ &\approx \mathcal{E}(\mathbf{r}_T, 0, t) \exp\left[i\frac{\omega}{c} n_0 z\right] \cdot \exp\left[i\frac{\omega}{2c} n_2 |\mathcal{E}(\mathbf{r}_T, 0, t)|^2 z\right]\end{aligned}$$

  - ▷ The nonlinear phase factor $\exp\left[i\omega n_2|\mathcal{E}(\mathbf{r}_T, 0, t)|^2 z/(2c)\right]$ varies in t and $\mathbf{r}_T$ because $|\mathcal{E}(\mathbf{r}_T, 0, t)|^2$ is proportional to the pulse's intensity
    - ∘ Results:
      - ◇ In fibers and bulk media: Self-phase modulation and self-steepening
      - ◇ In bulk media: Self-focusing

ELECTRIC POLARIZATION (6)

- Coupled-wave method for third-order nonlinear electric polarization:
  - ▷ Use a different field for each frequency and each orthogonal polarization:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \mathbf{E}_4$$

  - ▷ Assume the envelope forms

$$\mathbf{E}_l = \mathrm{Re}\left[\hat{\mathbf{e}}_l \psi_l(\mathbf{r}_T, \omega_l)\mathcal{E}_l(z,\, t)e^{i[\beta(\omega_l)z - \omega_l t]}\right]$$

and

$$\mathbf{P}_{NL} = \sum_n \mathrm{Re}\left[-i\hat{\mathbf{e}}_n \psi_n(\mathbf{r}_T, \omega_n)\mathcal{P}_n(z,\, t)e^{i[\beta(\omega_n)z - \omega_n t]}\right]$$

  - ▷ Electric fields at frequencies ω_1, ω_2, ω_3 generate polarization components at frequencies

$$\pm\omega_k, \quad \pm 3\omega_k, \quad \pm(2\omega_k - \omega_l), \quad \pm(\omega_k + \omega_l - \omega_m) \quad (k, l, m = 1, 2, 3)$$

ELECTRIC POLARIZATION (7)

- Single-field method for third-order nonlinear electric polarization:
  - ▷ Use one field, $\mathbf{E}$, for all WDM channels and all nonlinearly generated waves (Raman, Brillouin, etc.)
  - ▷ Assume the envelope form

$$\mathbf{E}(\mathbf{r}_T, z, t) = \mathrm{Re}\left[\hat{\mathbf{e}}_0 \psi(\mathbf{r}_T, \omega_0)\mathcal{E}(z,t)e^{i[\beta(\omega_0)z - \omega_0 t]}\right]$$

and

$$\mathbf{P}_{NL}(\mathbf{r}_T, z, t) = \mathrm{Re}\left[-i\hat{\mathbf{e}}_0 \psi(\mathbf{r}_T, \omega_0)\mathcal{P}(z,t)e^{i[\beta(\omega_0)z - \omega_0 t]}\right]$$

  - ▷ Required time step is shorter than for coupled-mode analysis, but the number of fields that must be computed and stored in memory is much smaller

FIBER NOTATION

Symbol	Identification	Units (cgs)
α	Attenuation coefficient	cm^{-1}
β	Propagation constant	cm^{-1}
β_1	$d\beta/d\omega$	$\mathrm{s\ cm}^{-1}$
β_2	$d^2\beta/d\omega^2$	$\mathrm{s}^2\ \mathrm{cm}^{-1}$
β_3	$d^3\beta/d\omega^3$	$\mathrm{s}^3\ \mathrm{cm}^{-1}$
$L_{\mathrm{eff}}(L)$	$(1 - e^{-\alpha L})/\alpha$	cm
L_D	$T_0^2/\lvert\beta_2\rvert$	cm
L_S	$T_0^3/\lvert\beta_3\rvert$	cm
L_{NL}	$(\gamma P_0)^{-1}$	cm
$\mathcal{E}$	Optical **E** field	$\mathrm{sV/cm}\ (=\mathrm{g}^{1/2}\mathrm{s}^{-3/2})$
$\mathcal{P}$	Electric polarization	$\mathrm{sV/cm}\ (=\mathrm{g}^{1/2}\mathrm{s}^{-3/2})$
$\chi^{(3)}$	3rd-order susceptibility	$(\mathrm{sV/cm})^{-3}\ (=\mathrm{g}^{-3/2}\mathrm{s}^{9/2})$