FOUR-WAVE MIXING (FWM) AND DENSE WDM

- Design problem: Maximize the number of channels in a WDM system, keeping the BER below a fixed value
  - ▷ If FWM did not exist, one could minimize BER by minimizing dispersion
  - ▷ Given 3 WDM channels at frequencies ω_1, ω_2, ω_3, FWM creates 9 new waves at frequencies

$$\omega_{ijk} = \omega_i + \omega_j - \omega_k \text{ where } i, j, k \in (1:3)$$

    - ∘ FWM is intermodulation distortion at optical frequencies
    - ∘ Result of FWM in a digital system is crosstalk
  - ▷ The only ways to minimize FWM are to:
    - ∘ Introduce dispersion into the system
    - ∘ Maximize channel separation ($\Rightarrow$ minimize number of channels)
  - ▷ **WDM system design depends on a compromise between dispersion control and FWM control**

NONLINEAR POLARIZATION FOR FWM (1)

- Nonlinear polarization:

$$\mathbf{P}_{NL} = 4\chi^{(3)}(\mathbf{E}\cdot\mathbf{E})\mathbf{E} = \sum_n \mathrm{Re}\left[-i\hat{\mathbf{e}}_n\psi_n(\mathbf{r}_T)\mathcal{P}_{NL,n}(z,\, t)e^{i(\beta_n z-\omega_n t)}\right]$$

$$= \tfrac{1}{2}\sum_n \left[-i\hat{\mathbf{e}}_n\psi_n\mathcal{P}_{NL,n}e^{i(\beta_n z-\omega_n t)} + \text{c.c.}\right]$$

The sum on n runs over all the frequencies that occur in $(\mathbf{E}\cdot\mathbf{E})\mathbf{E}$

- Assume three fields, with possibly different frequencies and polarizations, and possibly in different modes:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 = \sum_{l=1}^{3} \mathrm{Re}\left[\hat{\mathbf{e}}_l\psi_l(\mathbf{r}_T)\mathcal{E}_l(z,\, t)e^{i(\beta_l z-\omega_l t)}\right]$$

$$= \tfrac{1}{2}\sum_{l=1}^{3}\left[\hat{\mathbf{e}}_l\psi_l\mathcal{E}_l e^{i(\beta_l z-\omega_l t)} + \text{c.c.}\right]$$

NONLINEAR POLARIZATION FOR FWM (2)

- Expand $(\mathbf{E}\cdot\mathbf{E})\mathbf{E} = \mathbf{E}^2\mathbf{E}$:

$$
\begin{aligned}
&(\mathbf{E}\cdot\mathbf{E})\mathbf{E} \\
&= \left(\mathbf{E}_1^2 + \mathbf{E}_2^2 + \mathbf{E}_3^2 + 2\mathbf{E}_1\cdot\mathbf{E}_2 + 2\mathbf{E}_1\cdot\mathbf{E}_3 + 2\mathbf{E}_2\cdot\mathbf{E}_3\right)(\mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3) \\
&= \left[\mathbf{E}_1^2\mathbf{E}_1 + \mathbf{E}_2^2\mathbf{E}_2 + \mathbf{E}_3^2\mathbf{E}_3\right] \\
&\quad + \left[\left(\mathbf{E}_2^2 + \mathbf{E}_3^2\right)\mathbf{E}_1 + \left(\mathbf{E}_1^2 + \mathbf{E}_3^2\right)\mathbf{E}_2 + \left(\mathbf{E}_1^2 + \mathbf{E}_2^2\right)\mathbf{E}_3\right. \\
&\quad + (2\mathbf{E}_1\cdot\mathbf{E}_2)\mathbf{E}_2 + (2\mathbf{E}_1\cdot\mathbf{E}_3)\mathbf{E}_3 + (2\mathbf{E}_1\cdot\mathbf{E}_2)\mathbf{E}_1 \\
&\quad \left. + (2\mathbf{E}_2\cdot\mathbf{E}_3)\mathbf{E}_3 + (2\mathbf{E}_1\cdot\mathbf{E}_3)\mathbf{E}_1 + (2\mathbf{E}_2\cdot\mathbf{E}_3)\mathbf{E}_2\right] \\
&\quad + \left[(2\mathbf{E}_1\cdot\mathbf{E}_2)\mathbf{E}_3 + (2\mathbf{E}_1\cdot\mathbf{E}_3)\mathbf{E}_2 + (2\mathbf{E}_2\cdot\mathbf{E}_3)\mathbf{E}_1\right]
\end{aligned}
$$

- Each group of square-bracketed terms predicts different nonlinear-optical phenomena

EVALUATION OF $\mathbf{E}_1^2\mathbf{E}_1$

- Envelope form:

$$\mathbf{E}_1 = \tfrac{1}{2}\hat{\mathbf{e}}_1\left[W_1 + W_1^*\right]$$

where

$$W_1 = e^{i[\beta(\omega_1)z-\omega_1 t]}\psi_1(\mathbf{r}_T,\omega_1)\mathcal{E}_1(z,t)$$

- Then

$$\begin{aligned}\mathbf{E}_1^2\mathbf{E}_1 &= \tfrac{1}{8}\hat{\mathbf{e}}_1\left[W_1 + W_1^*\right]\left[W_1 + W_1^*\right]^2 \\ &= \tfrac{1}{8}\hat{\mathbf{e}}_1\left[W_1 + W_1^*\right]\left[W_1^2 + 2|W_1|^2 + W_1^{*2}\right] \\ &= \tfrac{1}{8}\hat{\mathbf{e}}_1\left[(W_1^3 + W_1^{*3}) + (W_1^2W_1^* + W_1^{*2}W_1) + 2|W_1|^2(W_1 + W_1^*)\right]\end{aligned}$$

  - ▷ Coefficient of third-harmonic terms at $\pm 3\omega_1$ is $\frac{1}{8}$
  - ▷ Coefficient of self-phase modulation terms at $\pm\omega_1$ is $\frac{3}{8}$

EVALUATION OF $\mathbf{E}_2^2\mathbf{E}_1$

- Envelope form:

$$\mathbf{E}_l = \tfrac{1}{2}\hat{\mathbf{e}}_l\,[W_l + W_l^*] \quad (l = 1, 2)$$

where

$$W_l = e^{i[\beta(\omega_l)z - \omega_l t]}\psi_l(\mathbf{r}_T, \omega_l)\mathcal{E}_l(z, t)$$

- Then

$$\begin{aligned}\mathbf{E}_2^2\mathbf{E}_1 &= \tfrac{1}{8}\hat{\mathbf{e}}_1\,[W_1 + W_1^*]\,[W_2 + W_2^*]^2 \\ &= \tfrac{1}{8}\hat{\mathbf{e}}_1\,[W_1 + W_1^*]\left[W_2^2 + 2|W_2|^2 + W_2^{*2}\right] \\ &= \tfrac{1}{8}\hat{\mathbf{e}}_1\left[W_1W_2^2 + W_1^*W_2^{*2}\right. \\ &\quad \left. + 2|W_2|^2(W_1 + W_1^*) + W_1^*W_2^2 + W_1W_2^{*2}\right]\end{aligned}$$

▷ Coefficient of FWM terms at $\pm(2\omega_2 - \omega_1)$ is $\frac{1}{8}$

▷ Coefficient of cross-phase modulation terms at $\pm\omega_1$ is $\frac{2}{8}$

EVALUATION OF $2(\mathbf{E}_1 \cdot \mathbf{E}_2)\mathbf{E}_2$

- Envelope form:

$$\mathbf{E}_l = \tfrac{1}{2}\hat{\mathbf{e}}_l \left[W_l + W_l^*\right] \quad (l = 1, 2)$$

where

$$W_l = e^{i[\beta(\omega_l)z - \omega_l t]}\psi_l(\mathbf{r}_T, \omega_l)\mathcal{E}_l(z, t)$$

- Then

$$\begin{aligned}
2(\mathbf{E}_1 \cdot \mathbf{E}_2)\mathbf{E}_2 &= \tfrac{2}{8}(\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2)\hat{\mathbf{e}}_2 \left[W_1 + W_1^*\right]\left[W_2 + W_2^*\right]^2 \\
&= \tfrac{2}{8}(\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2)\hat{\mathbf{e}}_2 \left[W_1 + W_1^*\right]\left[W_2^2 + 2|W_2|^2 + W_2^{*2}\right] \\
&= \tfrac{2}{8}(\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2)\hat{\mathbf{e}}_2 \Big[W_1 W_2^2 + W_1^* W_2^{*2} \\
&\qquad + 2|W_2|^2(W_1 + W_1^*) + W_1^* W_2^2 + W_1 W_2^{*2}\Big]
\end{aligned}$$

  - ▷ Coefficient of FWM terms at $\pm(2\omega_2 - \omega_1)$ is $\frac{2}{8}$
  - ▷ Coefficient of cross-phase modulation terms at $\pm\omega_1$ is $\frac{4}{8}$

NONLINEAR POLARIZATION FOR FWM (3)

- After substitution of the envelope form of $\mathbf{E}_l$, 3 kinds of terms occur in $(\mathbf{E} \cdot \mathbf{E})\mathbf{E}$:
  - ▷ General form: $\mathcal{E}_l^3$ or c.c.
    - ∘ Coefficient (same polarization): $\frac{1}{8}$
    - ∘ Frequencies: $\pm 3\omega_l$
    - ∘ Third-harmonic generation
  - ▷ General form: $|\mathcal{E}_l|^2\mathcal{E}_l$, $\mathcal{E}_k^2\mathcal{E}_l^*$ or c.c.
    - ∘ Coefficient (same polarization): $\frac{3}{8}$
    - ∘ Frequencies: $\pm\omega_l$ or $\pm(2\omega_k \pm \omega_l)$
    - ∘ SPM, degenerate FWM
  - ▷ General form: $|\mathcal{E}_k|^2\mathcal{E}_l$, $\mathcal{E}_k^*\mathcal{E}_l\mathcal{E}_m$ or c.c.
    - ∘ Coefficient (same polarization): $\frac{6}{8}$
    - ∘ Frequencies: $\pm\omega_l$, $\pm(\omega_k + \omega_l \pm \omega_m)$
    - ∘ XPM, non-degenerate FWM

NONLINEAR POLARIZATION FOR FWM (4)

- Compact formula for $(\mathbf{E}\cdot\mathbf{E})\mathbf{E}$:

$$\mathbf{E}^2\mathbf{E} = \frac{1}{8}\sum_{k,l,m} D_{|k|,|l|,|m|}\psi_k\psi_l\psi_m\mathcal{E}_k\mathcal{E}_l\mathcal{E}_m\, e^{i\{[\beta(\omega_k)+\beta(\omega_l)+\beta(\omega_m)]z-(\omega_k+\omega_l+\omega_m)t\}}$$

where $k,\ l,\ m = \pm 1,\ \pm 2,\ \pm 3,$

$$\mathcal{E}_{-|k|} = \mathcal{E}^*_{|k|},\quad \psi_{-|k|} = \psi^*_{|k|},\quad \beta_{-|k|} = -\beta_{|k|},\quad \omega_{-|k|} = -\omega_{|k|},$$

and, when all waves have the same polarization,

$$D_{|k|,|l|,|m|} = \begin{cases} 1, & \text{if } k = l = m; \\ 3, & \text{if } k = -l = m; \\ 6, & \text{if } \begin{cases} k = -l \neq m & \text{or if} \\ |k|, |l|, |m| & \text{are different.} \end{cases} \end{cases}$$

- Compact formula for $\mathbf{P}_{NL}$:

$$-\tfrac{1}{2} i\psi_n\mathcal{P}_n e^{i(\beta_n z-\omega_n t)} = \frac{1}{8}\left(4\chi^{(3)}\right) D_{|k|,|l|,|m|}\psi_k\psi_l\psi_m\mathcal{E}_k\mathcal{E}_l\mathcal{E}_m e^{i\{[\beta(\omega_k)+\beta(\omega_l)+\beta(\omega_m)]z-(\omega_k+\omega_l+\omega_m)t\}}$$

where $\omega_n = \omega_k + \omega_l + \omega_m$

NONLINEAR POLARIZATION FOR FWM (5)

- Multiply both sides of the equation for $\mathcal{P}_n$ by $\psi_n(\mathbf{r}_T)^*$ and integrate with respect to the transverse area:

$$\boxed{\mathcal{P}_n = i\chi^{(3)} D_{|k|,|l|,|m|} \mu_{nklm} \mathcal{E}_k \mathcal{E}_l \mathcal{E}_m e^{i(\Delta\beta_{nklm})z}}$$

where

$$\omega_n = \omega_k + \omega_l + \omega_m \, ,$$

the **wave-vector mismatch** is $\Delta\beta_{nklm} = -\beta(\omega_n) + \beta(\omega_k) + \beta(\omega_l) + \beta(\omega_m)$, and the **mode overlap integral** is

$$\mu_{nklm} = \frac{\iint \psi_n(\mathbf{r}_T)^* \psi_k(\mathbf{r}_T) \psi_l(\mathbf{r}_T) \psi_m(\mathbf{r}_T) \, d^2 r_T}{\iint |\psi_n(\mathbf{r}_T)|^2 \, d^2 r_T}$$

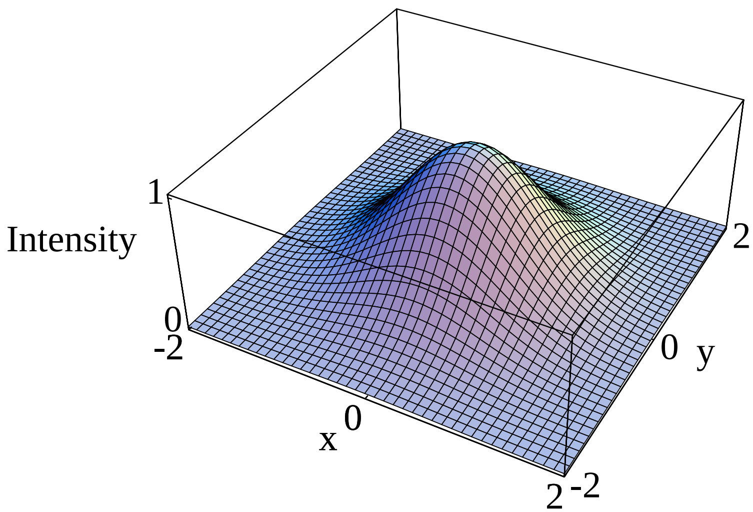
- Example: $\psi_n = \psi_k = \psi_l = \psi_m = \psi_1$

 ▷ ψ_1 is the eigenfunction for the HE_{11} mode,

$$\psi_1 \approx e^{-r^2/(2w^2)} \Rightarrow \mu_{nklm} = \frac{1}{2}$$

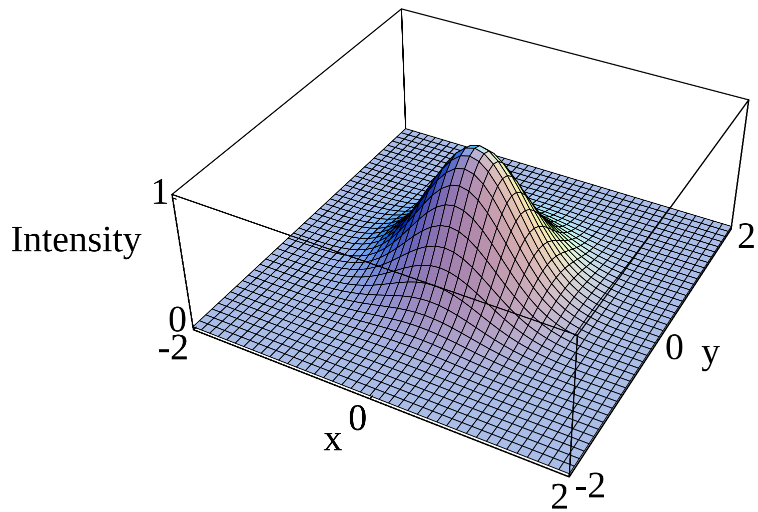
NONLINEAR POLARIZATION FOR FWM (6)

Intensity of lowest mode



The intensity in the lowest mode, $|\psi_1(x,\, y)|^2$

Mode overlap integrand



The mode overlap integrand, $|\psi_1(x,\, y)|^4$

NONLINEAR POLARIZATION FOR FWM (7)

- Mode overlap integral:

$$\mu_{nklm} = \frac{\iint \psi_n(\mathbf{r}_T)^* \psi_k(\mathbf{r}_T) \psi_l(\mathbf{r}_T) \psi_m(\mathbf{r}_T)\, d^2 r_T}{\iint |\psi_n(\mathbf{r}_T)|^2\, d^2 r_T}$$

  ▷ Normalization of ψ_n:

$$\iint |\psi_n(\mathbf{r}_T)|^2\, d^2 r_T = 1$$

  ▷ Then $|\psi_n(\mathbf{r}_T)|^2$ has units of $(\text{area})^{-1}$

- **Effective area** for the interaction of waves k, l, m to produce wave n:

$$(A_e)^{-1} := |\mu_{nklm}|$$

PROPAGATION EQUATION FOR FWM (1)

- Paraxial wave equation for the complex amplitude of a single mode:

$$\frac{\partial}{\partial z'}\overline{\mathcal{E}}_m = -\frac{2\pi\omega\sigma}{\beta c^2}\overline{\mathcal{E}}_m + \frac{2\pi\omega^2}{\beta c^2}\overline{\mathcal{P}}_{NL,m}$$

where

$$\begin{aligned}\mathbf{E} &= \mathrm{Re}\left[\hat{\mathbf{e}}\psi_m(\mathbf{r}_T)\mathcal{E}_m(z,\,t)e^{i(\beta z-\omega t)}\right]\\ &= \tfrac{1}{2}\left[\hat{\mathbf{e}}\psi_m\mathcal{E}_m e^{i(\beta z-\omega t)} + \text{c.c.}\right]\\ \mathbf{P}_{NL} &= \mathrm{Re}\left[-i\hat{\mathbf{e}}\psi_m(\mathbf{r}_T)\mathcal{P}_{NL,m}(z,\,t)e^{i(\beta z-\omega t)}\right]\\ &= \tfrac{1}{2}\left[-i\hat{\mathbf{e}}\psi_m\mathcal{P}_{NL,m}e^{i(\beta z-\omega t)} + \text{c.c.}\right]\end{aligned}$$

- Nonlinear polarization for four-wave mixing of a *linearly polarized* field in an isotropic material:

$$\mathbf{P}_{NL} = 4\chi^{(3)}(\mathbf{E}\cdot\mathbf{E})\mathbf{E}$$

PROPAGATION EQUATION FOR FWM (2)

- Paraxial wave equation for the complex amplitude of a single mode (or WDM channel):

$$\frac{\partial}{\partial z'}\overline{\mathcal{E}}_n = -\frac{\alpha}{2}\overline{\mathcal{E}}_n + \frac{2\pi i \omega_n^2 \chi^{(3)}}{\beta_n c^2}\mu_{nklm} D_{|k|,|l|,|m|}\overline{\mathcal{E}}_k \overline{\mathcal{E}}_l \overline{\mathcal{E}}_m e^{i(\Delta\beta_{nklm})z'}$$

- Introduce mode amplitudes which are directly related to power:

$$\mathcal{F}_n = \sqrt{\frac{c n_{0,n} A_e}{8\pi}}\mathcal{E}_n \;\Rightarrow\; \text{power in mode } n \text{ is } P_n = |\mathcal{F}_n|^2$$

- Paraxial wave equation in terms of $\mathcal{F}$'s:

$$\frac{\partial}{\partial z'}\overline{\mathcal{F}}_n = -\frac{\alpha}{2}\overline{\mathcal{F}}_n + i\frac{16\pi^2 \omega_n \chi^{(3)}}{n_{0,n}^2 c^2 A_e}\mu_{nklm} D_{|k|,|l|,|m|}\overline{\mathcal{F}}_k \overline{\mathcal{F}}_l \overline{\mathcal{F}}_m e^{i(\Delta\beta_{nklm})z'}$$

where we have used

$$\omega_n \approx \frac{c\beta_n}{n_{0,n}}$$

PROPAGATION EQUATION FOR FWM (3)

- Paraxial wave equation:

$$\frac{\partial}{\partial z'}\overline{\mathcal{F}}_n = -\frac{\alpha}{2}\overline{\mathcal{F}}_n + i\frac{16\pi^2\omega_n\chi^{(3)}}{n_{0,n}^2c^2A_e}\mu_{nklm}D_{|k|,|l|,|m|} \times \overline{\mathcal{F}}_k\overline{\mathcal{F}}_l\overline{\mathcal{F}}_m e^{i(\Delta\beta_{nklm})z'}$$

- Two regimes to study:
  - ▷ Three strong waves ($\mathcal{F}_k$, $\mathcal{F}_l$, $\mathcal{F}_m$) generate nine weak waves ($\mathcal{F}_n$)
    - ○ Useful for estimating crosstalk among channels
  - ▷ Parametric coupling of three strong waves
    - ○ Leads to coherent amplification of some channels and de-amplification of others

PARAMETRIC AMPLIFICATION BY FWM (1)

- Assume that $\omega_s \leq \omega_1 \leq \omega_a$,

$$\omega_l = \omega_k = \omega_1, \quad \omega_m = -\omega_a, \quad \omega_n = \omega_s = 2\omega_1 - \omega_a,$$

and that the wave at ω_1 is strong and undepleted

- **Parametric equations** for Stokes and anti-Stokes amplitudes:

$$\frac{\partial}{\partial z'}\overline{\mathcal{F}}_s = -\frac{\alpha}{2}\overline{\mathcal{F}}_s + i\frac{g}{2}\overline{\mathcal{F}}_a^* e^{i\Delta\beta_{as}z'}$$

$$\frac{\partial}{\partial z'}\overline{\mathcal{F}}_a^* = -\frac{\alpha}{2}\overline{\mathcal{F}}_a^* - i\frac{g}{2}\overline{\mathcal{F}}_s e^{-i\Delta\beta_{as}z'}$$

where

$$\Delta\beta_{as} = 2\beta(\omega_1) - [\beta(\omega_a) + \beta(\omega_s)],$$

$$D_{11a} = 3 \quad \text{and} \quad g = \frac{96\pi^2\omega_n\chi^{(3)}}{n_{0,n}^2 c^2 A_e}\mu_{s11a}P_1$$

PARAMETRIC AMPLIFICATION BY FWM (2)

- Solution of the parametric equations predicts exponential gain for the Stokes and anti-Stokes waves:

$$\overline{\mathcal{F}}_s(z',\, t') = -e^{-(\alpha + i\Delta\beta_{as})z'/2} \Big\{ \overline{\mathcal{F}}_s(0,\, t') \cosh(bz') + \frac{i}{2b} \left[\, g\overline{\mathcal{F}}_a^{\,*}(0,\, t') - i\Delta\beta_{as}\overline{\mathcal{F}}_s(0,\, t')\,\right] \sinh(bz') \Big\}$$

where

$$b = \tfrac{1}{2}\sqrt{g^2 - (\Delta\beta_{as})^2}$$

(& similarly for $\overline{\mathcal{F}}_a(z',\, t')$)

- **Threshold** for parametric gain:

$$g = \Delta\beta_{as}$$

- **Minimum channel spacing at threshold** (practical units):

$$\Delta\nu_{\text{eq,GHz}} = 11.65 \left[\frac{P_{1,\text{mW}}}{D_{\text{ps/nm-km}}} \right]^{1/2}$$

SPECTRUM OF FWM (1)

- Frequencies produced by the mixing of three strong waves at ω_1, ω_2 and ω_3:

$$\omega_{klm} = \omega_k + \omega_l - \omega_m$$

  - ▷ Case 1: $\{k, l, m\}$ = permutation of $\{1, 2, 3\}$ There are three initially weak waves with frequencies

$$\omega_{123}, \quad \omega_{231}, \quad \omega_{312}$$

$$\text{degeneracy factor } D_{|k|,|l|,|m|} = 6$$

  - ▷ Case 2: $\omega_l = \omega_k \neq \omega_m$. There are six initially weak waves with frequencies

$$\omega_{112}, \quad \omega_{113}, \quad \omega_{221}, \quad \omega_{223}, \quad \omega_{331}, \quad \omega_{332}$$

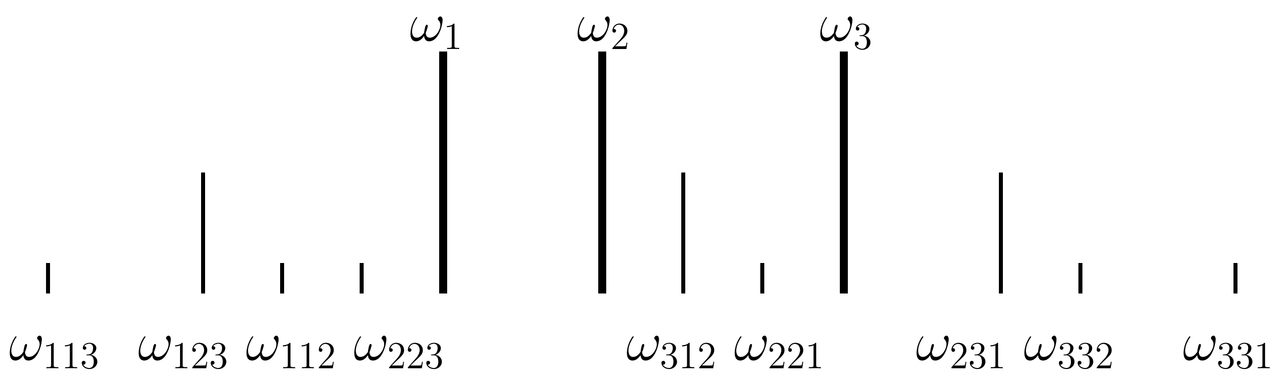
$$\text{degeneracy factor } D_{|k|,|k|,|m|} = 3$$

  - ▷ Intensity in Case 1 is 4 $\times$ intensity in Case 2

- Reference: N. Shibata *et al.*, *IEEE J. Quant. Elect.* **QE-23**, 1205–1210 (1987)

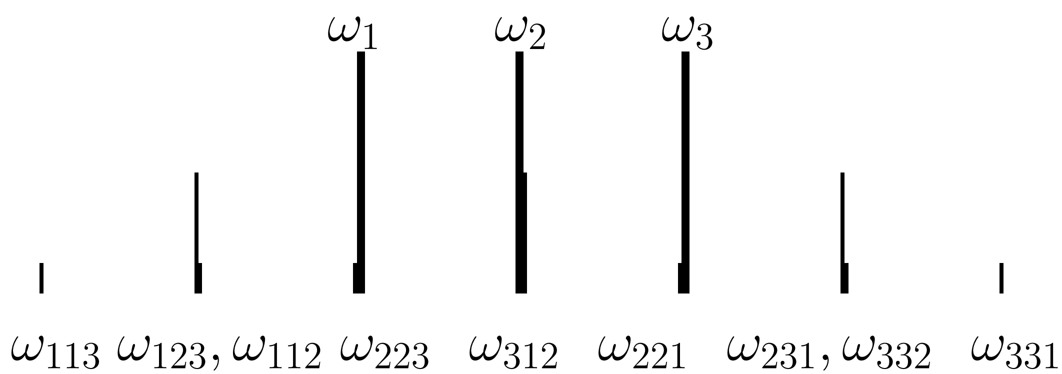
SPECTRUM OF FWM (2)

- Spectrum when $|\omega_1 - \omega_2| \neq |\omega_2 - \omega_3|$:



SPECTRUM OF FWM (3)

- Spectrum when $|\omega_1 - \omega_2| = |\omega_2 - \omega_3|$:



WEAK-WAVE GENERATION BY FWM (1)

- Assume that

$$\omega_1 = \omega_k, \quad \omega_2 = \omega_l, \quad \omega_3 = -\omega_m, \quad \omega_4 = \omega_n$$

and let

$$\Delta\beta = \Delta\beta_{nklm} = \beta(\omega_1) + \beta(\omega_2) - [\beta(\omega_3) + \beta(\omega_4)]$$

where $\omega_1 + \omega_2 - (\omega_3 + \omega_4) = 0$

- Paraxial wave equation for a weak-wave amplitude:

$$\frac{\partial}{\partial z'}\overline{\mathcal{F}}_4 = -\frac{\alpha}{2}\overline{\mathcal{F}}_4 + i\frac{\kappa}{L_e(L)}\overline{\mathcal{F}}_1\overline{\mathcal{F}}_2\overline{\mathcal{F}}_3^{\,*}e^{i(\Delta\beta)z'}$$

where

$$\kappa = \frac{16\pi^2\omega_n\chi^{(3)}L_e(L)}{n_{0,n}^2c^2A_e}\mu_{nklm}D_{|k|,|l|,|m|}$$

- Assume only linear attenuation of the strong waves:

$$\overline{\mathcal{F}}_i(z',\, t') = e^{-\alpha z'/2}\overline{\mathcal{F}}_i(0,\, t') \text{ for } i = 1,\, 2,\, 3$$

WEAK-WAVE GENERATION BY FWM (2)

- Transform away the exponential loss term:
  - ▷ Multiply
$$\frac{\partial}{\partial z'}\overline{\mathcal{F}}_4 = -\frac{\alpha}{2}\overline{\mathcal{F}}_4 + i\frac{\kappa}{L_e(L)}\overline{\mathcal{F}}_1\overline{\mathcal{F}}_2\overline{\mathcal{F}}_3^{*}e^{i(\Delta\beta)z'}$$
by $e^{\alpha z'/2}$ and introduce the temporary field
$$\mathcal{F}' = e^{\alpha z'/2}\,\overline{\mathcal{F}}_4$$
  - ▷ Assume linear attenuation of the strong waves:
$$\overline{\mathcal{F}}_i(z',\, t') = e^{-\alpha z'/2}\overline{\mathcal{F}}_i(0,\, t')$$
for $i = 1,\, 2,\, 3$
  - ▷ Phase mismatch:
$$\Delta\beta = \beta(\omega_1) + \beta(\omega_2) - [\beta(\omega_3) + \beta(\omega_4)]$$
  - ▷ Equation for $\mathcal{F}'$:
$$\frac{\partial}{\partial z'}\mathcal{F}' = i\frac{\kappa}{L_e(L)}e^{(i\Delta\beta-\alpha)z'}\overline{\mathcal{F}}_1(0)\overline{\mathcal{F}}_2(0)\overline{\mathcal{F}}_3^{*}(0)$$

WEAK-WAVE GENERATION BY FWM (3)

- Solution of the equation

$$\frac{\partial}{\partial z'}\mathcal{F}' = i\frac{\kappa}{L_e(L)} e^{(i\Delta\beta-\alpha)z'} \overline{\mathcal{F}}_1(0)\overline{\mathcal{F}}_2(0)\overline{\mathcal{F}}_3^*(0)$$

▷ Integrate:

$$\begin{aligned}\mathcal{F}'(L) &= i\kappa\overline{\mathcal{F}}_1(0)\overline{\mathcal{F}}_2(0)\overline{\mathcal{F}}_3^*(0)\int_0^L e^{(i\Delta\beta-\alpha)z'}\,dz' \\ &= \frac{1-e^{-(\alpha-i\Delta\beta)L}}{L_e(L)(\alpha-i\Delta\beta)}\kappa\overline{\mathcal{F}}_1(0)\overline{\mathcal{F}}_2(0)\overline{\mathcal{F}}_3^*(0)\end{aligned}$$

▷ Substitute $\mathcal{F}' = e^{\alpha z'/2}\,\overline{\mathcal{F}}_4$ and use $P_4(L) = |\overline{\mathcal{F}}_4(L)|^2$

▷ Power in the weak wave:

$$\boxed{|P_4(L)|^2 = \eta_{123}(\alpha,\Delta\beta,L)\,\kappa^2\,e^{-\alpha z'}\,P_1(0)P_2(0)P_3(0)}$$

where

$$\eta_{123}(\alpha,\Delta\beta,L) := \left|\frac{1-e^{-(\alpha-i\Delta\beta)L}}{L_e(L)(\alpha-i\Delta\beta)}\right|^2$$

WEAK-WAVE GENERATION BY FWM (4)

- Evaluation of

$$\eta_{123}(\alpha, \Delta\beta, L) = \left| \frac{1 - e^{-(\alpha - i\Delta\beta)L}}{L_e(L)(\alpha - i\Delta\beta)} \right|^2$$

using $L_e(L) = \alpha^{-1}(1 - e^{-\alpha L})$:

$$\left| \frac{1 - e^{-(\alpha - i\Delta\beta)L}}{L_e(L)(\alpha - i\Delta\beta)} \right|^2 = \frac{\alpha^2}{\alpha^2 + (\Delta\beta)^2} \frac{(1 - e^{-(\alpha - i\Delta\beta)L})(1 - e^{-(\alpha + i\Delta\beta)L})}{(1 - e^{-\alpha L})^2}$$

$$= \frac{\alpha^2}{\alpha^2 + (\Delta\beta)^2} \left[1 + \frac{2e^{-\alpha L}[1 - \cos(\Delta\beta L)]}{(1 - e^{-\alpha L})^2} \right]$$

$$= \frac{\alpha^2}{\alpha^2 + (\Delta\beta)^2} \left[1 + \frac{4e^{-\alpha L} \sin^2(\frac{1}{2}\Delta\beta L)}{(1 - e^{-\alpha L})^2} \right]$$

WEAK-WAVE GENERATION BY FWM (5)

- Weak-wave power at $z = L$:

$$\boxed{P_4(L) = \kappa^2\, e^{-\alpha L} \eta_{123}(\alpha, \Delta\beta, L) P_1(0) P_2(0) P_3(0)}$$

▷ The **phase mismatch factor**

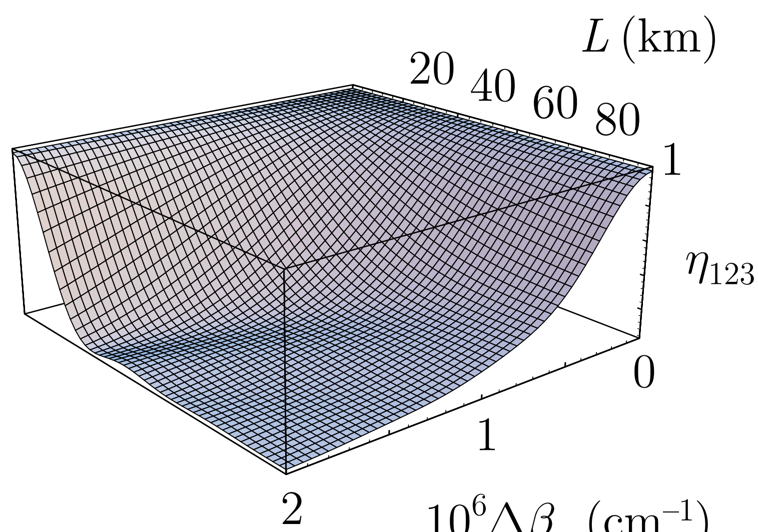
$$\eta_{123}(\alpha, \Delta\beta, L) = \frac{\alpha^2}{\alpha^2 + (\Delta\beta)^2} \times \left[1 + \frac{4e^{-\alpha L}\ \sin^2\left(\frac{1}{2}\Delta\beta L\right)}{\left(1 - e^{-\alpha L}\right)^2}\right]$$

is the ratio of the power generated without phase matching to the power generated with phase matching

EFFICIENCY OF FWM

- FWM efficiency at $\alpha = 0.2$ dB/km:

$$\eta_{123}(\alpha, \Delta\beta, L) = \frac{\alpha^2}{\alpha^2 + (\Delta\beta)^2}\left[1 + \frac{4e^{-\alpha L}\,\sin^2\left(\frac{1}{2}\Delta\beta L\right)}{\left(1 - e^{-\alpha L}\right)^2}\right]$$



WEAK-WAVE GENERATION BY FWM (6)

- Important special cases:
  - ▷ **Zero loss** ($\alpha = 0$):
    - ∘ The power in the weak wave is proportional to the reciprocal of the square of the propagation-constant mismatch:

$$P_4(L) \propto (\Delta\beta)^{-2}$$

    - ∘ $P_4(L)$ oscillates with a period equal to twice the nonlinear-optical coherence length,

$$l_{\mathrm{coh}} = \pi/\Delta\beta$$

  - ▷ **Long distance** ($L \gg \alpha^{-1}$):

$$P_4(L) \propto [1 + (\Delta\beta/\alpha)^2]^{-1}$$

WEAK-WAVE GENERATION BY FWM (7)

- Special case of **zero loss** ($\alpha = 0$):

$$\kappa^2 \eta_{123}(0,\, \Delta\beta,\, L) = \left(\frac{\kappa}{L}\right)^2 \frac{4\,\sin^2\left(\frac{1}{2}\Delta\beta L\right)}{(\Delta\beta)^2} = \kappa^2 \mathrm{sinc}^2\,\left(\tfrac{1}{2}\Delta\beta L\right)$$

  ▷ The phase mismatch of the waves radiated by $\mathbf{P}_{NL}(\omega_{123})$ at $z = 0$ and $z = L$ is $L\Delta\beta$

  ▷ If $\Delta\beta \neq 0$, the wave radiated by $\mathbf{P}_{NL}(\omega_{123})$ grows in amplitude from $z = 0$ to $z = l_{\mathrm{coh}}$, where the **nonlinear-optical coherence length** is

$$l_{\mathrm{coh}} = \frac{\pi}{\Delta\beta}$$

  At $z = l_{\mathrm{coh}}$, the sine takes its maximum value:

$$\sin\left(\tfrac{1}{2}\Delta\beta\, l_{\mathrm{coh}}\right) = 1$$

  ▷ $\mathbf{P}_{NL}(\omega_{123}) = 0$ at $z = 2l_{\mathrm{coh}}$, since

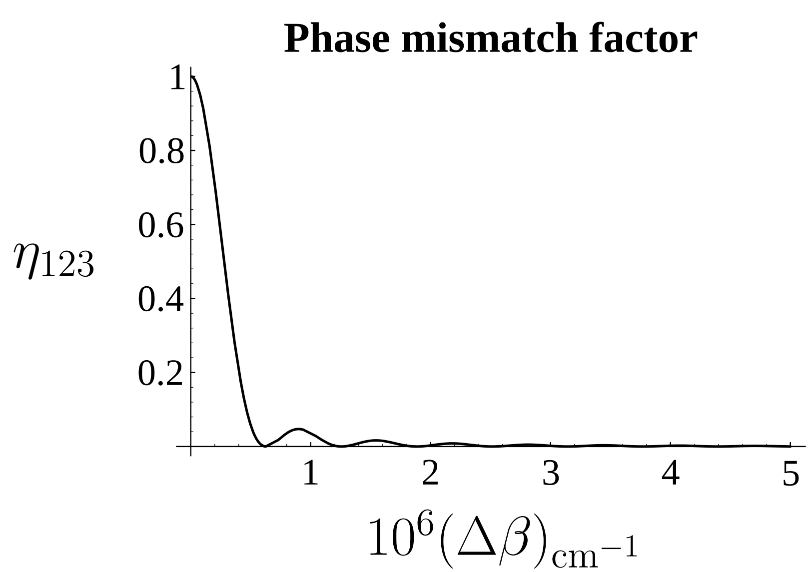
$$\sin\left(\Delta\beta\, l_{\mathrm{coh}}\right) = 0$$

WEAK-WAVE GENERATION BY FWM (8)

- **Zero-absorption limit**:

$$\eta_{123}(0, \Delta\beta, L) = \frac{4\sin^2(\frac{1}{2}\Delta\beta L)}{(\Delta\beta)^2}$$

- Plot for $\alpha = 0,\ L = 100$ km:



WEAK-WAVE GENERATION BY FWM (9)

- **Limit of zero dispersion** $(\Delta\beta = 0)$:
  - ▷ **Zero-loss limit** $(\alpha = 0)$:
    The wave at ω_4 grows in amplitude until the pump wave is depleted
  - ▷ **Non-zero loss** $(\alpha \neq 0)$:
    Ratio of the power $P_4(L)$ in the wave generated by FWM to the power $P_1(L) = e^{-\alpha L} P_1(0)$ in one of the original 3 waves is

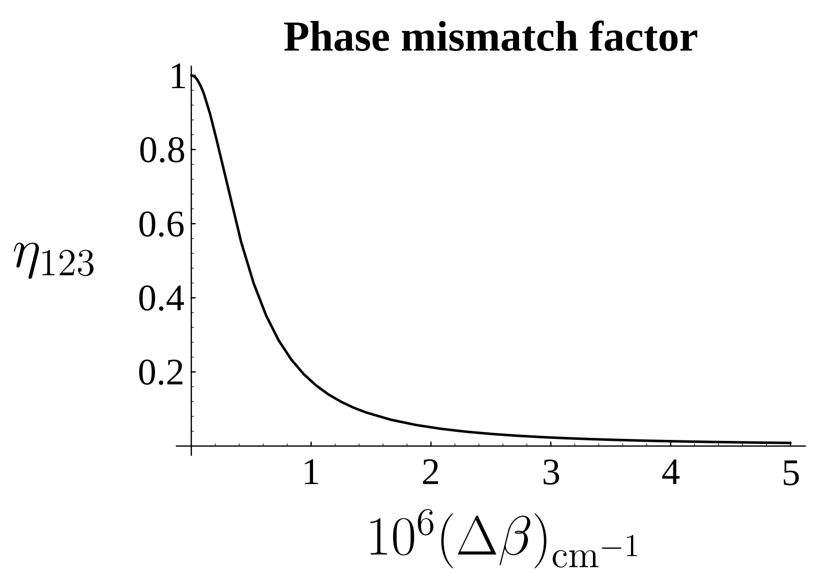
$$\frac{P_4(L)}{P_1(L)} = \kappa^2 \eta_{123}(\alpha, \Delta\beta, L) P_2(0) P_3(0)$$

  - ◦ The only dependence on distance is in the FWM efficiency, $\eta_{123}(\alpha, \Delta\beta, L)$
  - ◦ As L becomes large compared to an absorption length α^{-1}, $\eta_{123}(\alpha, \Delta\beta, L) \to 1$
  - ◦ **The only way to make FWM small when $\Delta\beta = 0$ is to decrease the launch power**

WEAK-WAVE GENERATION BY FWM (10)

- For **long-haul telecommunications**, where $e^{-\alpha L} \approx 0$:

$$\eta_{123}(\alpha,\, \Delta\beta,\, L) \approx \frac{1}{1 + \left(\dfrac{\Delta\beta}{\alpha}\right)^2}$$



- One can design for low FWM power by adjusting $\Delta\beta$

WEAK-WAVE GENERATION BY FWM (11)

- Express $\Delta\beta$ in terms of chromatic dispersion D and $S = dD/d\lambda$ in a form appropriate for ω_{312}:
  - ▷ Expand $\beta(\omega_r)$ $(r = 1,\ 3,\ 4)$ in a power series about ω_2
  - ▷ Let $\Delta\nu_{km} = |\nu_k - \nu_m|$, where $\omega_k = 2\pi\nu_k$:

$$\Delta\beta = \frac{2\pi\lambda^2}{c}\Delta\nu_{12}\Delta\nu_{23}\left[D + \frac{\lambda^2 S}{2c}(\Delta\nu_{12} + \Delta\nu_{23})\right]$$

  - ▷ Define an equivalent channel spacing

$$\Delta\nu_{\mathrm{eq}} = \sqrt{\Delta\nu_{12}\Delta\nu_{23}}$$

  - ▷ Assume equal channel spacing:

$$\Delta\nu_{12} = \Delta\nu_{23}$$

  - ▷ Then $\Delta\nu_{12}\Delta\nu_{23}(\Delta\nu_{12} + \Delta\nu_{23}) = 2\left(\Delta\nu_{\mathrm{eq}}\right)^3$
- Reference: R. G. Waarts *et al.*, *Proc. IEEE* **78**, 1344–1368 (1990)

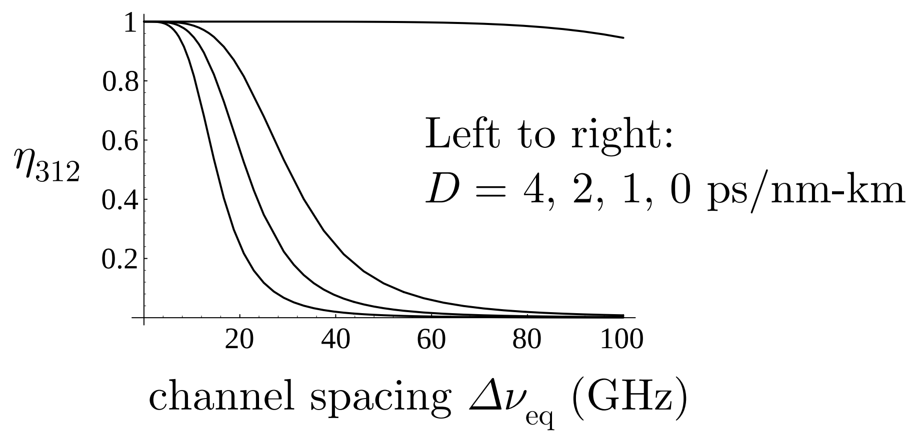
WEAK-WAVE GENERATION BY FWM (12)

- Plot the long-haul phase mismatch factor η_{312} versus $\Delta\nu_{\text{eq}}$ in terms of practical parameters:

$$\eta_{312}(\alpha,\ \Delta\beta,\ L) \approx \frac{1}{1+\left(\dfrac{\Delta\beta}{\alpha}\right)^2} \text{ where } \frac{\Delta\beta}{\alpha} = \frac{2\pi\lambda^2}{\alpha c}\Delta\nu_{\text{eq}}^2\left[D + \frac{\lambda^2 S}{2c}(2\Delta\nu_{\text{eq}})\right]$$

Phase mismatch factor

$\lambda = 1550$ nm, $S = .055$ ps/km-nm^2



WEAK-WAVE GENERATION BY FWM (13)

- Design problem: Minimize FWM crosstalk in a WDM system
  - ▷ Channel frequencies:
$$\nu_i = \nu_0 + n_i \Delta\nu$$
  - ▷ FWM frequencies:
$$\nu_{ijk} = \nu_0 + n_{ijk}\Delta\nu \quad \text{where} \quad n_{ijk} = n_i + n_j - n_k$$
  - ▷ Require that for all integers i, j, k from 1 to N such that $k \neq i, j$,
$$n_{ijk} \notin (n_1, n_2, \ldots, n_N) \Rightarrow \text{unequal channel spacings}$$
  - ▷ Require a minimum channel separation of $n\Delta\nu$:
$$n_{i+1} - n_i \geq n$$
  - ▷ Require minimum total optical bandwidth
$$B = (n_N - n_1)\Delta\nu = \text{minimum}$$
  - ▷ This integer linear programming problem is NP-complete!