Self-Induced Transparency (2-level system)

There are 2 cases: homogeneous & inhomogeneous broadening

A. Resonant case (homogeneous broadening) (undamped)

Bloch equations when $\Delta = 0$ (laser frequency = atomic freq.):

$$\frac{d\tilde{u}}{dt} = \Delta \tilde{v} = 0$$

$$\frac{d\tilde{v}}{dt} = -\Delta\tilde{u} + 2\Omega\tilde{w} = 2\Omega\tilde{w}$$

$$\frac{d\tilde{w}}{dt} = -2\Omega\tilde{v}$$

The eq^{n} $\frac{d\tilde{u}}{dt} = 0$ implies that $\tilde{u}(t) = \tilde{u}(-\infty)$.
We assume that the "initial" condition at $t = -\infty$ is

$$\boxed{\tilde{u}(-\infty) = 0 \Rightarrow \tilde{u}(t) = 0 \quad \text{(all } t)}$$

The equations $\frac{d\tilde{v}}{dt} = 2\Omega\tilde{w}$, $\frac{d\tilde{w}}{dt} = -2\Omega\tilde{v}$ $\quad [\Omega = \Omega(t)]$

with the initial condition $\tilde{w}(-\infty) = -1$, $\tilde{v}(-\infty) = 0$) can we define $\sin(-\infty)$?
(which is appropriate for an absorber) have the solution

$$\boxed{\begin{aligned} \tilde{v}(t) &= -\sin \Phi(t) \\ \tilde{w}(t) &= -\cos \Phi(t) \end{aligned}}$$

where $\boxed{\Phi(t) = 2\int_{-\infty}^{t} \Omega(t')\,dt'}$

We now put the equation for propagation of the electric field into a form that is useful for studying laser-pulse propagation. Maxwell's equations lead to

$$\left(\nabla^2 - \frac{\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \frac{4\pi}{c^2}\ddot{\vec{P}} \quad \text{Note that } \vec{E} \parallel \vec{P}.$$

if we neglect $\nabla(\nabla\cdot\vec{E})$. Introduce slowly varying amplitudes:

$$E = \tfrac{1}{2}\left(\mathcal{E} e^{+i(kz-\omega t)} + \mathcal{E}^* e^{-i(kz-\omega t)}\right)$$

$$P = \tfrac{1}{2}\left(\mathcal{P}' e^{i(kz-\omega t)} + \mathcal{P}'^* e^{-i(kz-\omega t)}\right)$$

where $k = \frac{\sqrt{\epsilon}\,\omega}{c} = \frac{n\omega}{c}$, $\quad n = \sqrt{\epsilon}$.

We can define $\mathcal{P}'$ as

$$\mathcal{P}' = 2\left[P e^{-i(kz-\omega t)}\right]_{SV}$$

where SV := slowly varying. In the usual way we get

$$\left[\nabla_T^2 + 2ik\left(\frac{\partial}{\partial z} + \frac{n}{c}\frac{\partial}{\partial t}\right)\right]\mathcal{E} = -\frac{4\pi\omega^2}{c^2}\mathcal{P}' = -\frac{4\pi k^2}{n^2}\mathcal{P}'.$$

Define the retarded time $\quad t' = t - nz/c$

$$z' = z$$

so that

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'}, \quad \frac{\partial}{\partial z} + \frac{n}{c}\frac{\partial}{\partial t} = \frac{\partial}{\partial z'} \quad \text{and } e^{i(kz-\omega t)} = e^{-i\omega t'}$$

If we can neglect the transverse Laplacian (e.g., if $\mathcal{E}$ is a plane wave), then

$$2ik\frac{\partial \mathcal{E}}{\partial z'} = -\frac{4\pi k^2}{n^2}\mathcal{P}' \Rightarrow \frac{\partial \mathcal{E}}{\partial z'} = \frac{2\pi i k}{n^2}\mathcal{P}'.$$

Now define
$$\mathcal{P} := i\mathcal{P}' = 2i\,[Pe^{i\omega t'}]_{sv}$$
so that
$$\boxed{\frac{\partial \mathcal{E}}{\partial z'} = \frac{2\pi k}{n^2}\mathcal{P}}\,.$$

Note that $\mathcal{E}$ and $\mathcal{P}$ are functions of z', t'.

We turn now to the problem of deriving an expression for $\mathcal{P}$ in terms of the Bloch-vector components of the density-matrix elements. We have
$$P = N\langle\mu\rangle = N\mu_{10}(c_0^* c_1 + c_0 c_1^*)$$
$$= N\mu_{10}(\tilde{c}_0^*\tilde{c}_1 e^{-i\omega t'} + \tilde{c}_0\tilde{c}_1^* e^{i\omega t'})$$

since $\frac{\partial}{\partial t} \to \frac{\partial}{\partial t'}$ in the time-dependent Schrödinger or density-matrix equations. Then
$$\mathcal{P} = 2i\,[Pe^{i\omega t'}]_{sv} = 2i N\mu_{10}\tilde{c}_0^*\tilde{c}_1$$
$$= iN\mu_{10}(\tilde{u} - i\tilde{v}) = N\mu_{10}(\tilde{v} + i\tilde{u}) \rightarrow \boxed{\mathcal{P} = N\mu_{10}(\tilde{v}+i\tilde{u})}$$

since $\tilde{u} = \tilde{c}_0^*\tilde{c}_1 + \tilde{c}_0\tilde{c}_1^*$, $\tilde{v} = i(\tilde{c}_0^*\tilde{c}_1 - \tilde{c}_0\tilde{c}_1^*)$.

For the propagation of a resonant pulse, $\tilde{u} = 0$.

Then
$$\boxed{\mathcal{P} = N\mu_{10}\tilde{v}}$$

where $\tilde{v}$ is real. Then $\partial\mathcal{E}/\partial z'$ is purely real:
$$\boxed{\frac{\partial \mathcal{E}}{\partial z'} = \frac{2\pi k N\mu_{10}}{n^2}\tilde{v} = -\frac{2\pi k\mu_{10}N}{n^2}\sin\Phi}\,. \qquad (*)$$

In general,

$$\mathcal{E} = |\mathcal{E}| e^{i\varphi}$$

where $\varphi = \varphi(z', t')$. But

$$\frac{\partial \mathcal{E}}{\partial z'} = \frac{\partial |\mathcal{E}|}{\partial z'} e^{i\varphi} + i |\mathcal{E}| e^{i\varphi} \frac{\partial \varphi}{\partial z'}$$

so that the imaginary part of $\partial \mathcal{E} / \partial z'$ vanishes iff

$$\frac{\partial \varphi}{\partial z'} = 0.$$

Then $\varphi = \varphi(t')$ only. If we assume that the incident pulse has $\varphi(t') = 0$, then $\mathcal{E}$ remains real as the field propagates.

We now find a real solution of (*) that represents a pulse propagating without a change of shape. For a pulse propagating without distortion in the $+z$ direction, with velocity V,

$$\boxed{\frac{\partial \mathcal{E}}{\partial z} = -\frac{1}{V} \frac{\partial \mathcal{E}}{\partial t}.} \qquad (**)$$

We assume that this equation and (*) both hold, and then derive the conditions that a distortionless solution of (*) must satisfy. In terms of z', t', (**) becomes

$$\frac{\partial \mathcal{E}}{\partial z} = \frac{\partial \mathcal{E}}{\partial z'} - \frac{n}{c} \frac{\partial \mathcal{E}}{\partial t'} = -\frac{1}{V} \frac{\partial \mathcal{E}}{\partial t'}$$

$$\Rightarrow \frac{\partial \mathcal{E}}{\partial z'} = -\frac{2\pi k N \mu_0}{n^2} \sin \Phi$$

$$= \left(\frac{n}{c} - \frac{1}{V} \right) \frac{\partial \mathcal{E}}{\partial t'}$$

$$\left.\begin{array}{l} \Rightarrow \boxed{\dfrac{\partial \mathcal{E}}{\partial t'} = \left(\dfrac{1}{V} - \dfrac{n}{c} \right)^{-1} \dfrac{2\pi k N \mu_{10}}{n^2} \sin \Phi} \\[2ex] \text{But } \dfrac{\partial \Phi}{\partial t'} = \dfrac{\partial \Phi}{\partial t} = \dfrac{\mu_{10} \mathcal{E}}{\hbar} \; (= 2\Omega) \Rightarrow \boxed{\mathcal{E} = \dfrac{\hbar}{\mu_{10}} \dfrac{\partial \Phi}{\partial t'}} \end{array}\right\} (\ddagger)$$

Plug this expression for $\mathcal{E}$ into our eq$^{\underline{n}}$ for $\partial\mathcal{E}/\partial t'$:

$$\frac{\hbar}{\mu_{10}}\frac{\partial^2\Phi}{\partial t'^2} = \left(\frac{1}{V}-\frac{n}{c}\right)^{-1}\frac{2\pi k N\mu_{10}}{n^2}\sin\Phi$$

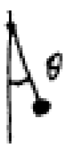
$$\Rightarrow \boxed{\frac{\partial^2\Phi}{\partial t'^2} = \frac{1}{\tau^2}\sin\Phi} \qquad (\dagger)$$

where

$$\tau^2 = \left[\left(\frac{1}{V}-\frac{n}{c}\right)^{-1}\frac{2\pi k N\mu_{10}^2}{\hbar n^2}\right]^{-1}.$$

Eq$^{\underline{n}}$ ($\dagger$) is (apart from a − sign) the same as the eq$^{\underline{n}}$ of motion of an undamped, finite-amplitude pendulum! However, we cannot just take over the solution from classical mechanics, because here we need to satisfy the initial condition

$$\Phi(t'=-\infty) = 0.$$



If we let θ be the usual pendulum angle, then

$$\theta = \pi - \Phi$$

so that the initial condition on θ is

$$\theta(-\infty) = \pi$$

The eq$^{\underline{n}}$ satisfied by θ is $\frac{\partial^2\theta}{\partial t^2} = -\frac{1}{\tau^2}\sin\theta$. Thus our problem corresponds to a pendulum that starts straight up.

Let's assume, by analogy with the physical pendulum, that

$$\tau^2 > 0.$$

From the eq^{n} defining τ^2, we have

$$\frac{2\pi k\, N\mu_{10}^2\, \tau^2}{\hbar n^2} = \frac{1}{V} - \frac{n}{c} > 0$$

$$\Rightarrow \qquad \boxed{V < \frac{c}{n}.}$$

The solution we seek, if it exists, therefore represents a pulse that travels more slowly than the phase velocity of light in the medium.

We now obtain the analytical form of $\mathcal{E}(z',t')$, starting from (‡), by multiplying the equation

$$\frac{\partial \mathcal{E}}{\partial t'} = \left(\frac{1}{V} - \frac{n}{c}\right)^{-1} \frac{2\pi k N\mu_{10}}{n^2} \sin\Phi$$

by the equation

$$\mathcal{E} = \frac{\hbar}{\mu_{10}} \frac{\partial \Phi}{\partial t'} :$$

Recall that the assumption of distortionless prop. implies that $\frac{\partial \mathcal{E}}{\partial z'} = \left(\frac{n}{c} - \frac{1}{V}\right)\frac{\partial \mathcal{E}}{\partial t'}$

(a standard method for deriving a conservation law)

$$\mathcal{E}\frac{\partial \mathcal{E}}{\partial t'} = \frac{1}{2}\frac{\partial}{\partial t'}\mathcal{E}^2 = \left(\frac{1}{V} - \frac{n}{c}\right)^{-1} \underbrace{\frac{2\pi k N\mu_{10}}{n^2}\frac{\hbar}{\mu_{10}}}_{\frac{2\pi k N\hbar}{n^2}} \sin\Phi \frac{\partial \Phi}{\partial t'}$$

$$= -\left(\frac{1}{V} - \frac{n}{c}\right)^{-1} \frac{2\pi k N\hbar}{n^2} \frac{\partial}{\partial t'}(\cos\Phi)$$

Then (with the initial condition)

$$\mathcal{E}^2 = 2\left(\frac{1}{V} - \frac{n}{c}\right)^{-1} \frac{2\pi N k\hbar}{n^2}(1 - \cos\Phi)$$

$$= 4\left(\frac{\hbar}{\mu_{10}}\right)^2 \frac{1}{\tau^2} \frac{1-\cos\Phi}{2}$$

$$\Rightarrow \boxed{\mathcal{E} = 2\frac{\hbar}{\mu_{10}\tau} \sin\frac{\Phi}{2}.} \qquad (\dagger\dagger)$$

This establishes the relationship that must exist between $\mathcal{E}$ and Φ if a real-amplitude pulse is to propagate without distortion.

Finally, we return to (‡) and solve for $\mathcal{E}$, using (††) along the way. We have (from (‡))

$$\frac{\partial \mathcal{E}}{\partial t'} = \frac{\hbar}{\mu_{10}\tau^2}\sin\Phi = \frac{\hbar}{\mu_{10}\tau^2}\cdot 2\sin\frac{\Phi}{2}\cos\frac{\Phi}{2}$$

from (††) $$= \frac{1}{\tau}\mathcal{E}\cos\frac{\Phi}{2} = \frac{1}{\tau}\mathcal{E}\sqrt{1-\sin^2\frac{\Phi}{2}}$$

(again) $$= \frac{1}{\tau}\mathcal{E}\left[1-\left(\frac{\mu_{10}\tau\mathcal{E}}{2\hbar}\right)^2\right]^{1/2}$$

This eq$^{\underline{n}}$ is separable:

$$\frac{dt'}{\tau} = \frac{d\mathcal{E}}{\mathcal{E}\left[1-\left(\frac{\mu_{10}\tau\mathcal{E}}{2\hbar}\right)^2\right]^{1/2}} = d\left[\operatorname{sech}^{-1}\left(\frac{\mu_{10}\tau\mathcal{E}}{2\hbar}\right)\right]$$

Then

$$\boxed{\mathcal{E} = \frac{2\hbar}{\mu_{10}\tau}\operatorname{sech}\frac{t'}{\tau}} \quad \Rightarrow \Omega = \frac{1}{\tau}\operatorname{sech}\frac{t'}{\tau} \quad \text{and} \quad \boxed{I \propto |\mathcal{E}|^2 \propto \operatorname{sech}^2\frac{t'}{\tau}}$$

A highly significant quantity is the <u>pulse area</u>

$$\boxed{\underset{=\Phi(\infty)}{\theta} := 2\int_{-\infty}^{\infty}\Omega(t')dt'} = \frac{2}{\tau}\int_{-\infty}^{\infty}\operatorname{sech}\left(\frac{t'}{\tau}\right)dt' = 2\int_{-\infty}^{\infty}\operatorname{sech} u\, du$$

$$\Rightarrow \theta = 2\cdot 2\tan^{-1}(e^u)\Big|_{-\infty}^{\infty} = 4(\tan^{-1}\infty - \tan^{-1}0)$$

$$= 4\cdot\frac{\pi}{2}$$

$$\Rightarrow \boxed{\theta = 2\pi}$$

This 2π solution (<u>2π pulse</u>) turns out to be the simplest solution. There are others; inverse-scattering theory shows that $2m\pi$ pulses break up into m 2π pulses. These pulses are true <u>solitons</u>, meaning solitary nonlinear waves that survive collisions. The important features are

(a) sech shape of $\mathcal{E}$
(b) $2m\pi$ area
(c) If $m > 1$, pulse breaks up into m 2π pulses
(d) true solitons
(e) For each integer m, index n, wavenumber k, dipole transition moment μ_{10}, & number density N, there are infinitely many solitons, one for each value of

$$\tau = \left[\left(\frac{1}{V} - \frac{n}{c} \right) \frac{\hbar\, n^2}{2\pi k N \mu_{10}^2} \right]^{1/2}$$

Each solution has its own speed V.
As $\tau \to 0$ (implying that $\mathcal{E} \to \infty$),

$$\frac{1}{V} \to \frac{n}{c}.$$

As $\tau \to \infty$ (implying that $\mathcal{E} \to 0$),

$$\frac{1}{V} \text{ becomes} \gg \frac{n}{c},$$

implying that $V \to 0$.

B. Resonant pumping, inhomogeneous broadening

Not all atoms have the same transition frequency if there are crystal fields, an atomic velocity distribution, etc. Let the probability distribution of detunings be $g(\Delta)$. Let the Bloch-vector components of an atom with detuning Δ be

$$\tilde{u}(\Delta,t'),\quad \tilde{v}(\Delta,t'),\quad \tilde{w}(\Delta,t').$$

The propagation equation is still

$$\frac{\partial \mathcal{E}}{\partial z'} = \frac{2\pi k}{n^2}\,\mathcal{P}$$

but now

$$\boxed{\mathcal{P} = N\mu_{10}\int_{-\infty}^{\infty}\left[\tilde{v}(\Delta,t') + i\tilde{u}(\Delta,t')\right]g(\Delta)\,d\Delta.}$$

We show that in the absence of damping, and if $\tilde{u}(t'=-\infty)=0$, $\tilde{u}$ is an odd function of Δ while $\tilde{v}$ and $\tilde{w}$ are even functions of Δ. The undamped Bloch equations are

$$\frac{d\tilde{u}}{dt} = \Delta\tilde{v}$$

$$\frac{d\tilde{v}}{dt} = -\Delta\tilde{u} + 2\Omega\tilde{w}$$

$$\frac{d\tilde{w}}{dt} = -2\Omega\tilde{v}$$

Let
$$\left.\begin{aligned} \Delta &\to \Delta' = -\Delta,\\ \tilde{u} &\to \tilde{u}' = -\tilde{u},\\ \tilde{v} &\to \tilde{v}' = \tilde{v},\\ \tilde{w} &\to \tilde{w}' = \tilde{w}. \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \frac{d\tilde{u}'}{dt} &= \Delta'\tilde{v}'\\ \frac{d\tilde{v}'}{dt} &= -\Delta'\tilde{u}' + 2\Omega\tilde{w}'\\ \frac{d\tilde{w}'}{dt} &= -2\Omega\tilde{v}'. \end{aligned}$$

If $\tilde{u}(t'=-\infty)=0$, then $\tilde{u}' = -\tilde{u}$ is a solution of the Bloch equations with $\Delta' = -\Delta$.

Now we assume that the detuning distribution g is symmetric (even):

$$g(-\Delta) = g(\Delta).$$

Then, since $\tilde{u}$ is odd and $\tilde{v}$ is even,

$$\mathcal{P} = N\mu_{10} \int_{-\infty}^{\infty} \tilde{v}(\Delta, t')\, g(\Delta)\, d\Delta.$$

Therefore $\mathcal{P}$ is real, implying that if $\mathcal{E}(t'=-\infty)$ is real, then $\mathcal{E}$ is real for all t' (see p. 4). Then

$$\frac{\partial \mathcal{E}}{\partial z'} = \frac{2\pi k}{n^2}\mathcal{P} = \frac{2\pi k N\mu_{10}}{n^2} \int_{-\infty}^{\infty} \tilde{v}(\Delta, t')\, g(\Delta)\, d\Delta.$$

It follows that the pulse area

$$\theta(z') = \frac{\mu_{10}}{\hbar} \int_{-\infty}^{\infty} \mathcal{E}(z', t')\, dt'$$

obeys the equation

$$\frac{\partial \theta}{\partial z'} = \frac{2\pi k N\mu_{10}^2}{\hbar n^2} \iint_{-\infty}^{\infty} \tilde{v}(\Delta, t')\, g(\Delta)\, d\Delta\, dt'.$$

In order to evaluate $\frac{\partial\theta}{\partial z'}$, we must make both assumptions about the detuning distribution g and approximations that will help us evaluate the integral. We assume that

$$|\Omega| \ll \text{width of } g$$

where the width is defined, for example, as $\langle(\Delta - \langle\Delta\rangle)^2\rangle^{1/2}$. Then, for most of the atoms,

$$|\Omega| \ll |\Delta|$$

so that the coherent driving by $\mathcal{E}$ is a weak perturbation on the oscillatory motion of $\tilde{u}$ and $\tilde{v}$. If we can neglect Ω compared to Δ, then the eq^{n}s satisfied by $\tilde{u}$ and $\tilde{v}$ are

$$\frac{d\tilde{u}}{dt} = \Delta\tilde{v}$$

$$\frac{d\tilde{v}}{dt} = -\Delta\tilde{u}.$$

The solution

$$\tilde{u}(\Delta,t') = \tilde{u}(\Delta,t_0')\cos\Delta(t'-t_0') + \tilde{v}(\Delta,t_0')\sin\Delta(t'-t_0')$$

$$\tilde{v}(\Delta,t') = -\tilde{u}(\Delta,t_0')\sin\Delta(t'-t_0') + \tilde{v}(\Delta,t_0')\cos\Delta(t'-t_0')$$

expresses $\tilde{u}$ and $\tilde{v}$ in terms of initial values at $t'=t_0'$. We make the approximation that these equations apply to all detunings, and substitute into the eq^n for $\partial\theta/\partial z'$:

$$\frac{\partial\theta}{\partial z'} \cong \frac{2\pi k N\mu_{10}^2}{\hbar n^2}\int_{-\infty}^{\infty} g(\Delta)\left\{\lim_{T\to\infty}\int_{-T+t_0'}^{T+t_0'}\left[-\tilde{u}(\Delta,t_0')\sin\Delta(t'-t_0') + \tilde{v}(\Delta,t_0')\cos\Delta(t'-t_0')\right]dt'\right\} d\Delta$$

The integral of $\tilde{u}(\Delta,t_0')\,g(\Delta)$ vanishes as before, leaving

$$\frac{\partial\theta}{\partial z'} \cong \frac{2\pi k N\mu_{10}^2}{\hbar n^2}\int_{-\infty}^{\infty} g(\Delta)\left\{\lim_{T\to\infty}\int_{-T+t_0'}^{T+t_0'}\tilde{v}(\Delta,t_0')\cos\Delta(t'-t_0')\,dt'\right\} d\Delta$$

$$\frac{\partial\theta}{\partial z'} \cong \frac{2\pi k N\mu_{10}^2}{\hbar n^2}\int_{-\infty}^{\infty} g(\Delta)\left\{\lim_{T\to\infty}\frac{[\sin(\Delta T)-\sin(-\Delta T)]}{\Delta}\right\}\tilde{v}(\Delta,t_0')\,d\Delta$$

$$\cong \frac{4\pi k N\mu_{10}^2}{\hbar n^2}\lim_{T\to\infty}\int_{-\infty}^{\infty} g(\Delta)\,\frac{\sin\Delta T}{\Delta}\,\tilde{v}(\Delta,t_0')\,d\Delta$$

As T becomes large, the function $\frac{\sin\Delta T}{\Delta}$ becomes more & more sharply peaked in Δ, approaching a const. times a δ-distribution as $T\to\infty$. The area is

$$\int_{-\infty}^{\infty}\frac{\sin(\Delta T)}{\Delta}\,d\Delta = \int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx \qquad (x=\Delta T)$$

$$= \frac{1}{2i}\lim_{\gamma\to 0}\int_{-\infty}^{\infty}\frac{e^{ix}-e^{-ix}}{x}\,e^{-\gamma x^2}\,dx$$

$$= \frac{2}{2i}\lim_{\gamma\to 0}\int_{-\infty}^{\infty}\frac{e^{ix-\gamma x^2}}{x}\,dx$$

$$= 2\pi.$$

Then $$\frac{\sin(\Delta T)}{\Delta}\ \xrightarrow[T\to\infty]{}\ 2\pi\,\delta(\Delta)$$

Therefore

$$\frac{\partial\theta}{\partial z'} \cong \frac{8\pi^2 k N\mu_{10}^2}{\hbar n^2}\,g(0)\,\tilde{v}(0,t_0') = \frac{8\pi^2 k N\mu_{10}^2}{\hbar n^2}\,g(0)\overbrace{\left[-\sin\Phi(t_0')\right]}^{\tilde{v}(t_0')\text{ for }\Delta=0}$$

We are free to let $t_0'\to\infty$; then $\Phi(t_0')\xrightarrow[t_0'\to\infty]{}\theta$, and

$$\frac{\partial\theta}{\partial z'} \cong -\frac{8\pi^2 k N\mu_{10}^2}{\hbar n^2}\,g(0)\sin\theta$$

$$\Rightarrow \boxed{\frac{\partial\theta}{\partial z'} = -\frac{\alpha}{2}\sin\theta} \qquad \text{where} \quad \alpha = \frac{(4\pi)^2 k N\mu_{10}^2}{\hbar n^2}\,g(0)$$

This equation implies the remarkable result that whether a pulse is absorbed or transmitted depends on the initial pulse area.

(a) Small area, $\theta \ll 1$. Then

$$\frac{\partial \theta}{\partial z'} \cong -\frac{\alpha}{2}\theta \Rightarrow \theta(z') = \theta(0)\, e^{-\frac{\alpha}{2}z'}$$

which is Beers' law (exponential attenuation)

(b) General area. The eq^{n} is separable:

$$\frac{d\theta}{\sin\theta} = -\frac{\alpha}{2}\,dz'$$

$$\Rightarrow \int_{\theta(0)}^{\theta(z')} \frac{d\theta}{\sin\theta} = -\frac{\alpha}{2}z'$$

(Now make the usual ½-angle substitution)

$$= \int_{\theta(0)}^{\theta(z')} \frac{d\theta}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \int_{\theta(0)}^{\theta(z)} \frac{\sec^2\frac{\theta}{2}}{2\tan\frac{\theta}{2}}\,d\theta$$

let $u = \tan\frac{\theta}{2}$

$\Rightarrow du = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta$

$$= \int_{\tan\theta(0)}^{\tan\theta(z')} \frac{du}{u} = \ln\left[\frac{\tan\frac{\theta(z')}{2}}{\tan\frac{\theta(0)}{2}}\right]$$

$$\Rightarrow e^{-\frac{\alpha}{2}z'} = \frac{\tan\frac{\theta(z')}{2}}{\tan\frac{\theta(0)}{2}}$$

$$\Rightarrow \boxed{\tan\frac{\theta(z')}{2} = e^{-\frac{\alpha}{2}z'}\tan\frac{\theta(0)}{2}}$$

The behavior predicted by this eqⁿ depends (in a discontinuous way) on the value of $\theta(0)$.

(i) If $0 < \frac{\theta(0)}{2} < \frac{\pi}{2}$, ie if $\boxed{0 < \theta(0) < \pi}$, then

$\infty > \tan\frac{\theta(0)}{2} > 0$

$\Rightarrow \quad \infty > \tan\frac{\theta(z')}{2} > 0 \quad \Rightarrow \quad \boxed{0 < \theta(z') < \pi}$

Moreover, $\tan\frac{\theta(z')}{2} \xrightarrow[\alpha z' \to \infty]{} 0 \quad \Rightarrow \quad \boxed{\theta(z') \xrightarrow[\alpha z' \to \infty]{} 0}$.

In this case the pulse is absorbed.

(ii) If $\frac{\pi}{2} < \frac{\theta(0)}{2} < \pi$, ie if $\boxed{\pi < \theta(0) < 2\pi}$, then

$-\infty < \tan\frac{\theta(0)}{2} < 0 \quad \Rightarrow \quad -\infty < \tan\frac{\theta(z')}{2} < 0 \quad \Rightarrow \quad \boxed{\pi < \theta(z') < 2\pi}$.

Now the fact that $\tan\frac{\theta(z')}{2} \xrightarrow[\alpha z' \to \infty]{} 0$ implies that

$$\boxed{\theta(z') \xrightarrow[\alpha z' \to \infty]{} 2\pi}.$$

The pulse is not absorbed at all!
[This is an example of a <u>bifurcation</u>.]

(iii) If $\pi < \frac{\theta(0)}{2} < \frac{3\pi}{2}$, ie if $\boxed{2\pi < \theta(0) < 3\pi}$,

then $\boxed{\theta(z') \xrightarrow[\alpha z' \to \infty]{} 2\pi}$.

(iv) If $\frac{3\pi}{2} < \frac{\theta(0)}{2} < 2\pi$, ie if $\boxed{3\pi < \theta(0) < 4\pi}$,

then $\boxed{\theta(z') \xrightarrow[\alpha z' \to \infty]{} 4\pi}$ and so on.

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