2) Self-focusing

We consider effects due to the nonlinear susceptibility

$$\chi' \cong \chi^{(1)\prime} + \chi^{(3)\prime} \langle E(t)^2 \rangle_t$$

or, where $|\chi^{(1)\prime}| \ll 1$, the nonlinear index

$$n' \cong n_0 + n_2 \langle E(t)^2 \rangle_t \, .$$

If the laser intensity varies as a function of the (transverse) distance from the axis of the beam, then a nonlinear index of refraction predicts the formation of a "lens" that is positive or negative depending on whether $n_2 > 0$ or $n_2 < 0$. If the beam propagates a distance L, then the optical path at finite intensity will be

$$\Delta p(E_0) = n' \frac{\omega}{c} L \cong \frac{\omega}{c}\left(n_0 + \tfrac{1}{2} n_2 E_0^2\right) L$$

while a beam at zero intensity would experience an optical path

$$\Delta p(0) = n_0 \frac{\omega}{c} L .$$

The difference,

$$\Delta p(E_0) - \Delta p(0) \cong \tfrac{1}{2} n_2 E_0^2 L \frac{\omega}{c}$$

indicates how much the center of a Gaussian beam is retarded (if $n_2>0$) or advanced (if $n_2<0$), and indicates a corresponding curvature of the wavefront. In terms of the transverse distance r, suppose that

$$E_0(r) = E_0(0)\, e^{-r^2/w_0^2}$$

so that $\Delta\phi(r) = \frac{\omega}{c} n_0 L + \frac{1}{2} n_2 E_0(0)^2 \frac{L\omega}{c} e^{-r^2/w_0^2}$.

Near $r=0$, expand:

$$\Delta\phi(r) \cong \left(\frac{\omega}{c} n_0 L + \frac{1}{2} n_2 E_0^2 L \frac{\omega}{c}\right) - \underbrace{\frac{1}{2} n_2 E_0^2 \frac{\omega}{c} L \frac{r^2}{w_0^2}}_{}$$

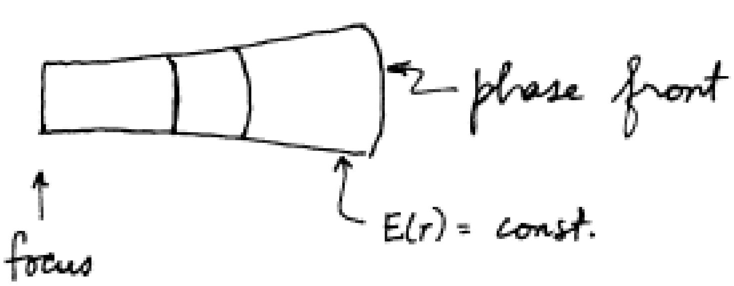
$$\cong \Delta\phi(0) - k_{NL} L$$

phase advance of edges with respect to center

where $k_{NL} = \frac{1}{2} n_2 E_0^2 \frac{\omega}{c} \frac{r^2}{w_0^2}$.

We expect that a focusing instability will occur when $n_2>0$ and when E_0^2 is sufficiently large that the phase advance (in r) exactly balances the phase retardation of the edges due to diffraction.

For a Gaussian beam:



The complex field amplitude of a Gaussian beam in a linear medium with index n_0 is

$$\mathcal{E}(r) = \mathcal{E}(0) \exp i \left\{ \frac{k r^2}{2q(z)} + P(z) \right\}$$

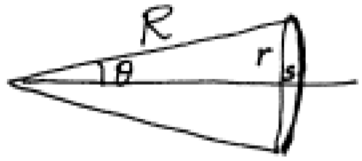
where $$\frac{1}{q(z)} = \frac{z + i\frac{kw_0^2}{2}}{z^2 + \left(\frac{kw_0^2}{2}\right)^2} \cdot \quad \left(= \frac{1}{z - i\frac{kw_0^2}{2}} \right)$$

and $k = n_0 \frac{\omega}{c}$.

When $|z| \ll \frac{kw_0^2}{2}$ = confocal parameter = length of focal region,

$$\frac{kr^2}{2q(z)} \cong \underbrace{\frac{z\,kr^2}{2\left(\frac{kw_0^2}{2}\right)^2}}_{\substack{\text{phase change}\\ \text{in } r\text{: wavefront}\\ \text{curvature}}} + \underbrace{\frac{i\,kr^2}{2\left(\frac{kw_0^2}{2}\right)}}_{\substack{\text{Gaussian amplitude}\\ \text{dependence}}} .$$

Recall that in the sagittal approximation,



$$s = R - R\cos\theta = R(1-\cos\theta) \cong R\frac{\theta^2}{2}$$

but $r = R\sin\theta \cong R\theta$, so $s \cong \frac{r^2}{2R}$

Thus $ks \cong \frac{kr^2}{2R} \cong \frac{kz r^2}{2\left(\frac{kw_0^2}{2}\right)^2}$, so $R \cong \left(\frac{kw_0^2}{2}\right)^2 \cdot \frac{1}{z}$.

Summary:

phase retardation of edges due to diffraction

$$\cong \frac{k r^2 z}{2\left(\frac{k w_0^2}{2}\right)^2}$$

phase advance due to nonlinear index

$$\cong \frac{1}{2} n_2 E_0^2 \frac{\omega}{c} \frac{r^2}{w_0^2} z = \frac{1}{2} \frac{n_2}{n_0} E_0^2 k \frac{r^2}{w_0^2} z .$$

These are equal when

$$\frac{1}{2} \frac{n_2}{n_0} E_0^2 k \frac{r^2}{w_0^2} z = \frac{k r^2 z}{2\left(\frac{k w_0^2}{2}\right)^2}$$

ie when $\boxed{E_0^2 w_0^2 = \frac{4 n_0}{n_2 k^2} .}$

Since $E_0^2 w_0^2$ is proportional to the total power in a Gaussian beam, we have shown that there is a <u>critical power for self-focusing</u>. In this phenomenon of <u>whole-beam self-focusing</u>, the entire beam collapses when the power exceeds the critical power because the wavefront distortion due to self-focusing exceeds that due to diffraction & is of the opposite sign.

Evaluation of critical power P_c:

The total power in the beam is

$$\frac{cE_0^2}{8\pi}\int_0^\infty e^{-2r^2/w_0^2}\cdot 2\pi r\,dr = \frac{cE_0^2}{8\pi}\cdot 2\pi w_0^2\int_0^\infty e^{-2u^2}\,u\,du$$

$$= \frac{cE_0^2}{4}w_0^2\cdot\frac{1}{2}\int_0^\infty e^{-u^2}\,d(u^2) = \frac{cE_0^2 w_0^2}{16}$$

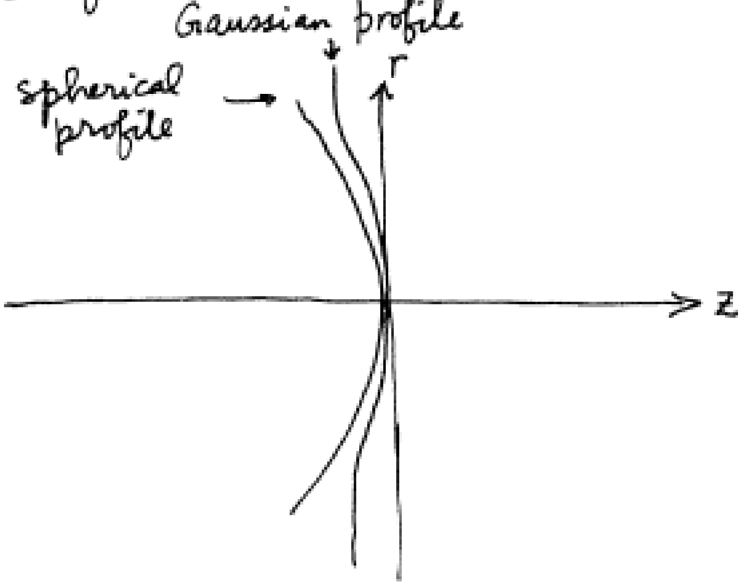
so $$\boxed{P_c = \frac{c n_0}{4 n_2 k^2}}$$

Critique of the usual derivations of whole-beam self-focusing:

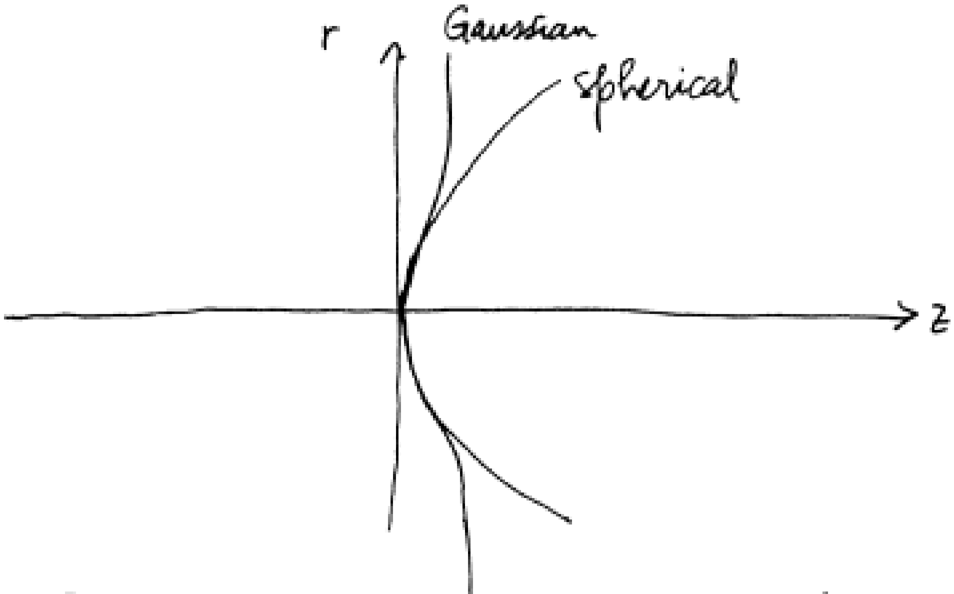
The factor $\exp i\left\{\frac{kr^2}{2q(z)} + \Delta\phi(r)\right\}$

is, as we shall see, the correct lowest-order expression for the propagation of a Gaussian beam in the presence of a nonlinear index. However, the expansion we have made in r^2 to find the critical power omits a very important detail: the wavefront curvature produced by diffraction is reasonably spherical, while that produced by SF is eminently non-spherical except at the very center of the beam.

Self-focusing alone

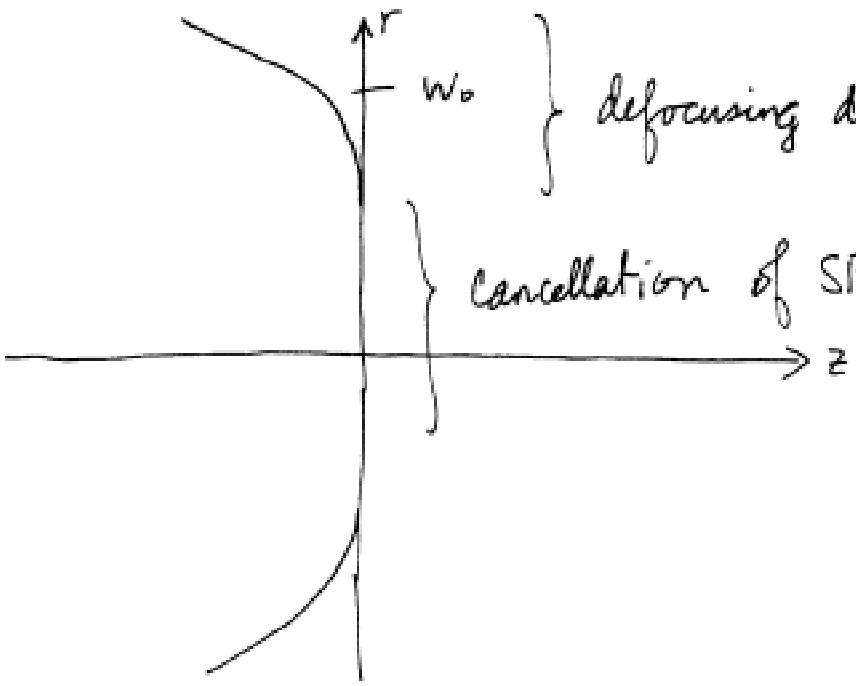


$n_2 < 0$



$n_2 > 0$

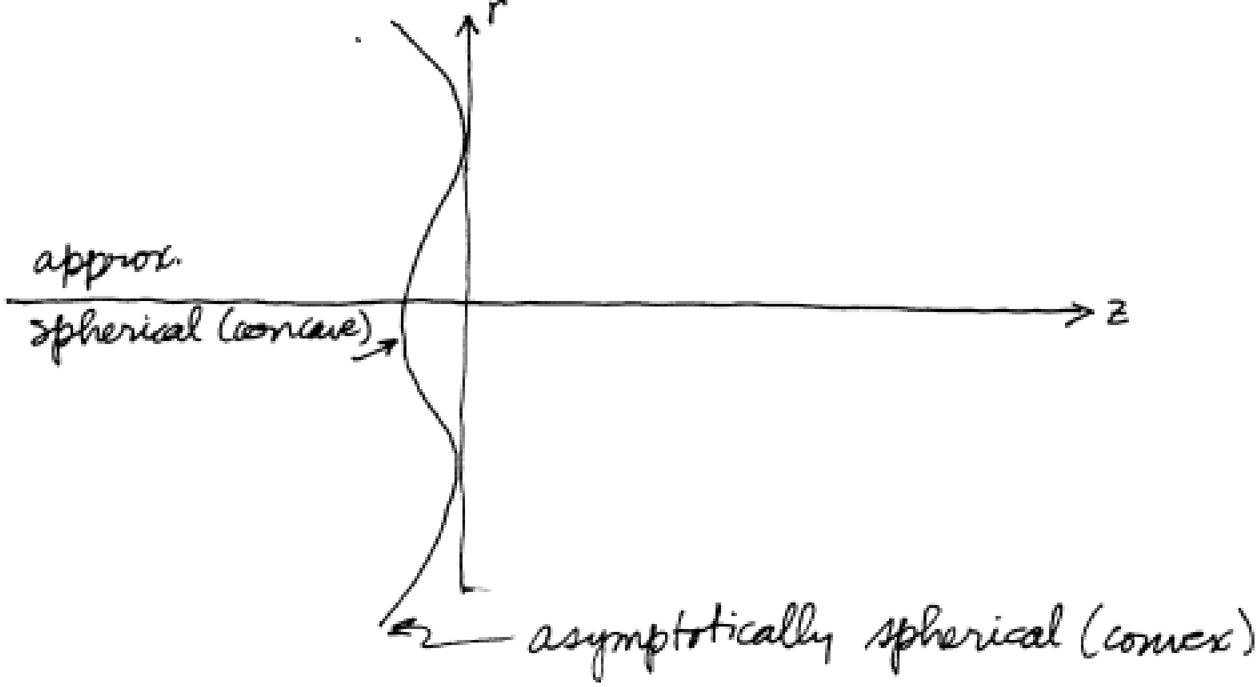
Result of SF and diffraction (for $P = P_c$):



defocusing due to diffraction

cancellation of SF & diffraction

Result of SF and diffraction (for $P > P_c$):



Propagation of a laser field in the paraxial-ray approximation

Objective: description of propagation including transverse effects

We use as the dipole density

$$\vec{P} = \chi^{(1)'}\vec{E} + \chi^{(NL)}\vec{E}$$

and take $\epsilon^{(1)'} = 1 + 4\pi\chi^{(1)'}$, $\epsilon^{(NL)} = 4\pi\chi^{(NL)'}$.
From Maxwell's equations we find

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t}\left[(\epsilon^{(1)'} + \epsilon^{(NL)})\vec{E}\right]$$

so $\nabla \times (\nabla \times \vec{E}) = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[(\epsilon^{(1)'} + \epsilon^{(NL)})\vec{E}\right]$

We neglect $\frac{\partial^2}{\partial t^2}\epsilon^{(NL)}$, even though $\epsilon^{(NL)}$ depends on $|\vec{E}|$. We further neglect $\nabla(\nabla\cdot\vec{E})$ (for justification, see Lax, Louisell & McKnight). We take

$$\epsilon^{(1)'} = n_0^2$$

Then

$$\nabla^2\vec{E} - \frac{1}{c^2}\left[n_0^2\frac{\partial^2\vec{E}}{\partial t^2} + \epsilon^{(NL)}\frac{\partial^2\vec{E}}{\partial t^2}\right] = 0$$

Let $\vec{E} = Re\left(\mathcal{E}\, e^{i(kz-\omega t)}\right)\hat{e}$

unit vector; polarization

where $k = n_0\omega/c$.
Then, with $\nabla^2 = \frac{\partial^2}{\partial z^2} + \nabla_T^2$, we find

$$\left[\nabla^2 - \frac{1}{c^2}\left(n_0^2 + \epsilon^{(NL)}\right)\frac{\partial^2}{\partial t^2}\right]\mathcal{E}\, e^{i(kz-\omega t)}$$

$$= \left\{\left(\frac{\partial^2}{\partial z^2} + \nabla_T^2\right)\mathcal{E} + 2ik\frac{\partial \mathcal{E}}{\partial z} - k^2\mathcal{E}\right.$$
$$\left. - \frac{1}{c^2}\left(n_0^2 + \epsilon^{(NL)}\right)\left[\frac{\partial^2 \mathcal{E}}{\partial t^2} - 2i\omega\frac{\partial \mathcal{E}}{\partial t} - \omega^2\mathcal{E}\right]\right\} e^{i(kz-\omega t)}$$

$$\cong \left\{\nabla_T^2\mathcal{E} + 2ik\frac{\partial \mathcal{E}}{\partial z} + 2ik\frac{n_0}{c}\frac{\partial \mathcal{E}}{\partial t} + \frac{\omega^2}{c^2}\epsilon^{(NL)}\mathcal{E}\right\} e^{i(kz-\omega t)}$$

so that

$$\left[\nabla_T^2 + 2ik\left(\frac{\partial}{\partial z} + \frac{n_0}{c}\frac{\partial}{\partial t}\right) + \frac{k^2}{n_0^2}\epsilon^{(NL)}\right]\mathcal{E} = 0.$$

Change variables to eliminate the time dependence:

$z' = z$	$z = z'$
$t' = t - n_0 z/c$	$t = t' + \frac{n_0}{c} z'$

If $f(z,t) = \bar{f}(z,t')$, then

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial z}dz = \frac{\partial \bar{f}}{\partial t'}dt' + \frac{\partial \bar{f}}{\partial z}dz$$

$$= \frac{\partial f}{\partial t}\left(dt' + \frac{n_0}{c}dz'\right) + \frac{\partial f}{\partial z}dz'$$

$$= \left(\frac{\partial f}{\partial z} + \frac{n_0}{c}\frac{\partial f}{\partial t}\right)dz' + \frac{\partial f}{\partial t}dt'$$

Then $\boxed{\frac{\partial \bar{f}}{\partial z'} = \frac{\partial f}{\partial z} + \frac{n_0}{c}\frac{\partial f}{\partial t}.}$

From now on we regard $\mathcal{E}$ as a function of x, y, z' and t', rather than x, y, z and t.

Then

$$\boxed{\left[\nabla_T^2 + 2ik\frac{\partial}{\partial z'} + \frac{k^2}{n_0^2}\epsilon^{(NL)}\right]\mathcal{E}(\vec{r}_T, z', t') = 0.}$$

Beam breakup due to self-focusing

We know that

$$\begin{aligned}
\epsilon = \quad & \epsilon^{(NL)} + \epsilon^{(1)\prime} = n^2 \\
= \; & (n_0 + n_{NL})^2 \\
= \; & n_0^2 + 2n_0 n_{NL} + n_{NL}^2 \\
\cong \; & n_0^2 + 2n_0 n_{NL}
\end{aligned}$$

so that $\boxed{\epsilon^{(NL)} \cong 2n_0 n_{NL}}$

In the usual approximation for a 2-level system,

$$\begin{aligned}
n_{NL} \cong \; & n_2 \langle E(t)^2 \rangle_t \\
\cong \; & \tfrac{1}{2} n_2 |\mathcal{E}(\vec{r}, t)|^2
\end{aligned}$$

Then the paraxial eq$^{\underline{n}}$ becomes

$$\boxed{\left[\nabla_T^2 + 2ik\frac{\partial}{\partial z} + \gamma|\mathcal{E}|^2\right]\mathcal{E} = 0}$$

a kind of nonlinear Schrödinger eq$^{\underline{n}}$ in which $\frac{\partial^2}{\partial t^2} \to \nabla_T^2$. Thus SF is a spatial version of soliton formation & pulse compression.

where $\gamma|\mathcal{E}|^2 = \frac{k^2}{n_0^2}\epsilon^{(NL)} = \frac{k^2}{n_0^2}\cdot 2n_0 n_{NL}$

$$= \frac{2k^2 n_{NL}}{n_0} = \frac{2k^2\cdot\frac{1}{2}n_2|\mathcal{E}|^2}{n_0}$$

so $\gamma = \frac{k^2 n_2}{n_0}$.

We now rewrite the paraxial eq$^{\underline{n}}$ in terms of amplitude & phase variables:

let $A = |\mathcal{E}(\vec{r},t')|$; $\quad \mathcal{E}(\vec{r},t') = A(\vec{r},t')\,e^{i\varphi(\vec{r},t')}$

$$\nabla_T^2(Ae^{i\varphi}) = \nabla_T\cdot\nabla_T(Ae^{i\varphi})$$

$$= \nabla_T\cdot\left[e^{i\varphi}\nabla_T A + iAe^{i\varphi}\nabla_T\varphi\right]$$

$$= e^{i\varphi}\nabla_T^2 A + 2ie^{i\varphi}\left[\nabla_T A\cdot\nabla_T\varphi\right]$$
$$+ (i)^2 Ae^{i\varphi}\left[\nabla_T\varphi\right]^2 + iAe^{i\varphi}\nabla_T^2\varphi$$

$$2ik\frac{\partial}{\partial z}(Ae^{i\varphi}) = 2ik\left[e^{i\varphi}\frac{\partial A}{\partial z} + ie^{i\varphi}A\frac{\partial\varphi}{\partial z}\right]$$

Then

$$e^{i\varphi}\nabla_T^2 A + 2i\, e^{i\varphi}[\nabla_T A \cdot \nabla_T \varphi] - A e^{i\varphi}[\nabla_T \varphi]^2 + iA e^{i\varphi}\nabla_T^2 \varphi$$
$$+ 2ik\left[e^{i\varphi}\frac{\partial A}{\partial z} + i e^{i\varphi} A \frac{\partial \varphi}{\partial z}\right] + \gamma A^3 e^{i\varphi} \qquad = 0$$

Divide by $e^{i\varphi}$ and equate the real & imaginary parts of the expression to zero separately:

real part:

$$\boxed{\nabla_T^2 A \quad - A[\nabla_T \varphi]^2 - 2kA\frac{\partial \varphi}{\partial z} + \gamma A^3 \quad = 0} \qquad (1)$$

imaginary part:

$$\boxed{2\nabla_T A \cdot \nabla_T \varphi + A\nabla_T^2 \varphi + 2k\frac{\partial A}{\partial z} = 0} \qquad (2)$$

The general solution of these two coupled nonlinear equations is quite difficult, even numerically. We therefore proceed by successive approximations.

First look for a <u>plane-wave solution</u>, where

$\nabla_T A = \nabla_T \varphi = 0$ so that $\nabla_T^2 A = \nabla_T^2 \varphi = 0$.

Then from (2) we find

$$\frac{\partial A}{\partial z} = 0 \quad \Rightarrow \quad \boxed{\begin{array}{l} A(\vec{r}, t') = A_0(t'), \\ \text{indep. of } x, y, z. \end{array}}$$

As we have seen in our qualitative discussion, a beam with a transverse peak in intensity becomes more focused and hence experiences a still higher peak intensity. We thus suspect an <u>instability</u>: a nonzero growth rate for a perturbation in the transverse dependence of the amplitude & phase. To investigate this possibility we employ a standard method for problems in which the Laplacian (or transverse Laplacian) enters the equations.

From (1) we find that

$$-2kA_0\frac{\partial\varphi}{\partial z'} + \gamma A_0^3 = 0$$

so that $$\frac{\partial\varphi}{\partial z'} = \frac{\gamma}{2k}A_0^2 \Rightarrow \varphi(\vec{r},t') = \varphi(0,0,0,t') + \frac{\gamma}{2k}A_0^2(0,0,0,t')\,z'$$

ie $$\boxed{\varphi^{(0)} = \varphi_0 + \frac{\gamma}{2k}A_0^2 z'}$$

This makes good sense physically, because we have already seen that

$$\frac{\gamma}{2k}A_0^2 z' = \frac{kn_2}{2n_0}E_0^2(t')z'$$

is the increase in optical path of the wavefront due to the nonlinear index of refraction.

The lowest-order solution is therefore

$$\boxed{\begin{aligned}\mathcal{E}^{(0)}(\vec{r},t') &= E_0 \exp\left(i\frac{kn_2}{2n_0}E_0^2(t')z'\right)\\ E(\vec{r},t') &= \mathrm{Re}\left[\mathcal{E}^{(0)}(\vec{r},t')\,e^{-i\omega t'}\right]\end{aligned}} \quad -i\omega t' = i(kz-\omega t)$$

We thus obtain the expression for the field that is the point of departure for our qualitative discussion of whole-beam self-focusing, provided we make the additional approximation of treating each piece of wavefront like a plane wave.

To obtain the next order of approximation, consider a perturbation

$$A = A_0 \left[1 + u(z') \psi_{\vec{q}}(\vec{r}_T) \right]$$

$$\varphi = \varphi^{(0)} + \tilde{\varphi}(z') \psi_{\vec{q}}(\vec{r}_T)$$

Recall that Eq. (2) implies that φ has z and $\vec{r}_T$ dependences.

where $\vec{r}_T = (x, y)$, and where we assume at the outset that ψ_q satisfies the eq$^{\underline{n}}$

$$\boxed{\nabla_T^2 \psi_{\vec{q}} = -q^2 \psi_{\vec{q}}}$$

so that ψ_q is a function that oscillates in the xy plane with a total spatial frequency q. For example,

$$\psi_{\vec{q}} = \cos(q_x x) \quad \cos(q_y y)$$

where $\vec{q} = (q_x, q_y)$.

If u and $\tilde{\varphi}$ are small, we can neglect their product and squares. This permits us to linearize the problem & obtain coupled linear equations for $\varphi^{(1)}$ and u.

Now $\nabla_T A = u(z') \nabla_T \psi_{\vec{q}}(\vec{r}_T)$

and $\nabla_T \varphi = \tilde{\varphi}(z') \nabla_T \psi_{\vec{q}}(\vec{r}_T)$

so that

$2\nabla_T A \cdot \nabla_T \varphi = 2u(z') \tilde{\varphi}(z) [\nabla_T \psi_{\vec{q}}]^2$
& will be neglected to this order of approximation.

From (2) we find that

$$A_0 \tilde{\varphi} \nabla_T^2 \psi_{\vec{q}} + 2kA_0 \frac{du}{dz'} \psi_{\vec{q}} = 0$$

$$= -q^2 A_0 \tilde{\varphi} \psi_{\vec{q}} + 2kA_0 \frac{du}{dz'} \psi_{\vec{q}}$$

so that $$\boxed{2k \frac{du}{dz'} = q^2 \tilde{\varphi}(z')} \quad (3)$$

From (1) we find

$$A_0 u \nabla^2 \psi_{\vec{q}} - 2kA_0 (1 + u\psi_q) \left[\frac{d\varphi^{(0)}}{dz'} + \frac{d\tilde{\varphi}}{dz'} \psi_q \right]$$
$$+ \gamma A_0^3 [1 + u\psi_q]^3 = 0$$

which simplifies to (using $\frac{d\varphi^{(0)}}{dz'} = \frac{\gamma}{2k} A_0^2$)

$$-q^2 u \psi_q - 2k \left[\frac{\gamma}{2k} A_0^2 + u\psi_q \frac{\gamma}{2k} A_0^2 + \frac{d\tilde{\varphi}}{dz'} \psi_q \right]$$
$$+ \gamma A_0^2 + 3\gamma A_0^2 u \psi_q = 0$$

when u and $\tilde{\varphi}$ are both small. We then have

$$\boxed{\left(-q^2 + 2\gamma A_0^2\right) u(z') = 2k \frac{d\tilde{\varphi}(z)}{dz'}} \qquad (4)$$

We now eliminate $\tilde{\varphi}$.
Differentiate (3) w. resp. to z and substitute in (4):

$$\frac{d\tilde{\varphi}}{dz'} = \frac{2k}{q^2} \frac{d^2u}{dz'^2}$$

$$\left(-q^2 + 2\gamma A_0^2\right) u = \frac{(2k)^2}{q^2} \frac{d^2u}{dz'^2}$$

or $$\boxed{\frac{d^2u}{dz'^2} = \frac{q^2}{4k^2}\left(-q^2 + 2\gamma A_0^2\right) u}$$

Let $\boxed{u = \delta e^{\alpha z'}}$. Then

$$\alpha^2 = \frac{q^2}{4k^2}\left(2\gamma A_0^2 - q^2\right)$$

$$\boxed{\alpha = \pm \frac{|\vec{q}|}{2k}\left(2\gamma A_0^2 - q^2\right)^{1/2}}$$

When $\boxed{2\gamma A_0^2 = \frac{2k^2 n_2 E_0^2}{n_0} > q^2}$

the amplitude of the mode $\vec{q}$ grows exponentially in z.

Since the size of a transverse "bump" is inversely proportional to $|\vec{q}|$, the criterion

$$\frac{2k^2 n_2 E_0^2}{n_0} > q^2$$

is essentially the same physically as the criterion

$$\frac{k^2 n_2 E_0^2}{4n_0} > \frac{1}{w_0^2}$$

found previously, and amounts to requiring that the increase in wavefront concavity due to self-focusing exceed the convexity due to diffraction. Evidently this condition puts a lower limit on the spatial size of a feature that can grow exponentially. The maximum $|\vec{q}|$ for which exponential growth can occur is

$$|\vec{q}|_{max} = \sqrt{2\gamma}\, A_0 .$$

Let us now find the behavior of the phase front by obtaining $\tilde{\varphi}(z)$: From (3),

$$\tilde{\varphi}(z) = \frac{2k}{q^2}\frac{du}{dz} = \frac{2k}{q^2}\alpha\, u(z)$$

To find the mode $\forall \vec{q}$ with maximum growth rate, we look for an α such that $\frac{d\alpha}{dq^2} = 0$. It is more convenient, and equivalent, to find

$$\frac{d\alpha^2}{dq^2} = 0 = \frac{1}{4k^2}\left(2\gamma A_0^2 - q^2\right) - \frac{q^2}{4k^2}$$

ie $$\boxed{\vec{q}_M^{\,2} = \gamma A_0^2}$$

The modes with the maximum growth rate

$$\pm \alpha_M = \frac{\sqrt{\gamma A_0^2}}{2k}\left[2\gamma A_0^2 - \gamma A_0^2\right]^{1/2}$$

$$= \frac{|\gamma A_0^2|}{2k}$$

will soon outstrip all the other modes, and will be essentially all that is seen in the transverse dependence of the amplitude & phase. We then have

$$\boxed{\begin{aligned} A(x,y,z',t') &\cong A_0(t')\left[1 + \delta e^{\alpha_M z'} \cos(q_x x)\cos(q_y y)\right] \\ \varphi(x,y,z',t') &\cong \varphi^{(0)}(z') + \delta e^{\alpha_M z'} \cos(q_x x)\cos(q_y y) \end{aligned}}$$

where $q_x^2 + q_y^2 = q^2 = \gamma A_0^2$.

In fact, the actual mode pattern will be a sum over q_x, q_y. This pattern can be studied experimentally by imposing an amplitude & phase variation on the incident beam, and then looking at the amplification of this pattern. In that case only the incident mode is important, and a simple pattern results: If $q_x = q_y = \frac{q}{\sqrt{2}}$, the intensity will be

$\updownarrow$ spacing $= \frac{\pi}{|q_y|} = \frac{\sqrt{2}\,\pi}{q_m} = \Delta y$

Area of one cell $= \Delta x \Delta y$

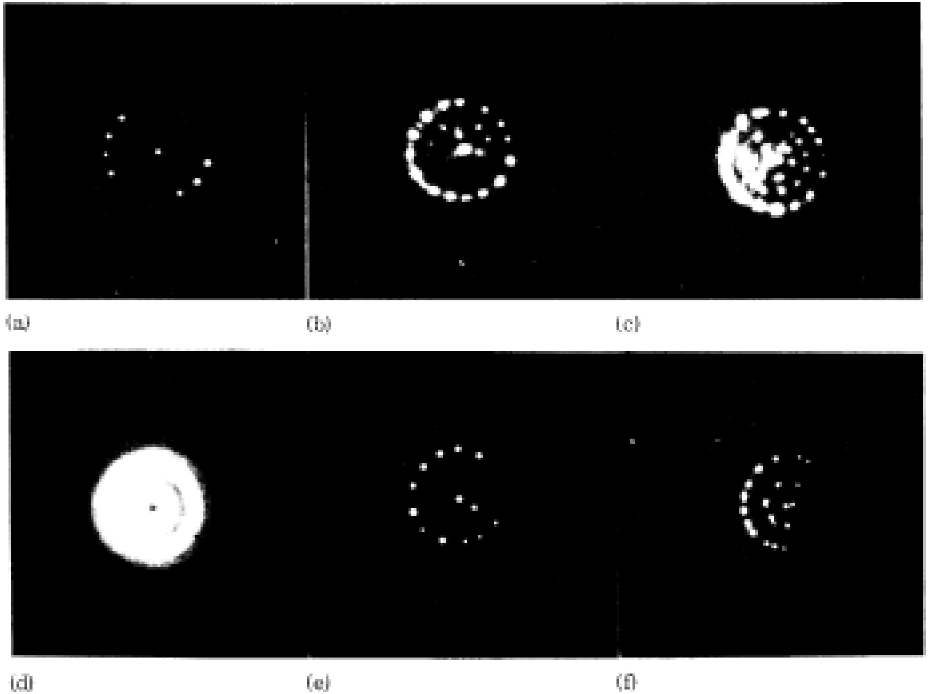
$$= \frac{2\pi^2}{q_m^2} = \frac{2\pi^2}{\gamma A_0^2}$$

The intensity is $\frac{c}{8\pi} A_0^2$ plus a small perturbation; then the power per cell is

$$\frac{c}{8\pi} A_0^2 \,\Delta x \Delta y = \frac{c}{8\pi} E_0^2 \cdot \frac{2\pi^2}{\frac{k^2 n_2}{n_0} E_0^2}$$

$$\boxed{P_{BU} \simeq \frac{\pi c n_0}{4 k^2 n_2} = \pi P_c}$$

That is, the power per cell under beam breakup is π times the critical power for whole-beam self-focusing.



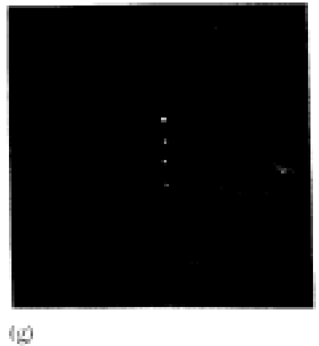


FIG. 2. Experimental photographs showing spatial periodicity of focal spots: (a), (b), and (c) show focal spot patterns for $F=7$ for progressively higher beam powers; (d) is the 6943-Å diffraction pattern of the end of the cell for $F=4$, while (e) and (f) are the corresponding focal spot patterns [(f) forced into aperiodicity by asymmetric intensity distribution)]; (g) is the focal spot pattern for a straight edge diffraction pattern.

metrized by placing the aperture slightly off axis of the Gaussian intensity distribution, one side of the ring pattern self-focuses earlier than the other side. For this case the spot spacing becomes intensity dependent, as shown in Fig. 2(f), with the foci becoming more closely spaced on the more intense side. Each cell again contains approximately four critical powers. At powers much higher than threshold (> 30%), some of the hot spots become out of focus at the end of the cell (in fact, hot spots in one annular ring can be in focus, while in another they can be out of focus), and in some cases diffraction interference phenomena are observed between light originating from different focal spots.

In Fig. 2(g) the focal spot pattern corresponding to straight edge diffraction is shown. Again, the focal spots appear at about equal intervals, but along straight lines. Similar patterns were observed for diffraction by a slit.

Our experimental results can be explained by an extension of an instability theory first suggested by Bespalov and Talanov.[5] This theory treats self-focusing as an instability phenomenon, wherein an initial perturbation (e.g., dust, refractive index inhomogeneities, etc.), no matter how small, grows to what are ultimately catastrophic proportions. So long as the perturbation remains small, it is accurately described by the usual linearized perturbation theory, whose solution is usually expressed as a complete set of normal modes, each with its associated growth rate. Among these normal modes, some simple subset will have a maximum growth rate, and hence after a short time, these will dominate everything else. This subset of normal modes with maximum growth rate establishes a cell pattern which then persists, each cell ultimately producing one self-focal spot. On this basis, then, a certain regularity of the focal spot pattern is not surprising. It has been shown theoretically[6] that exactly one critical power goes into each focus, but additional background power is required to focus in a finite distance. Thus, each cell contains a few critical powers. This can be seen more quantitatively in a couple of examples.

First let us consider the total collapse of a beam of initial Gaussian profile

$$E(r, 0) = E_0 \exp(-r^2/2a^2). \tag{1}$$

The phenomenon is governed by the well known quasi-optical equation[2]

$$\frac{2i\omega}{v}\frac{\partial E}{\partial z} = \nabla_\perp^2 E + \gamma |E|^2 E, \tag{2}$$

where we have written γ in place of $3\omega^2\epsilon_2/4c^2$, v represents the phase velocity of light in the medium, ω is the frequency of the light, and ϵ_2 is the nonlinear susceptibility coefficient. Moment theory applied to Eq. (2) gives the self-focusing length as[7]

$$z_f = \omega a^2 / v(\tfrac{1}{4}\gamma E_0^2 a^2 - 1)^{1/2}. \tag{3}$$

For the Gaussian, critical power is given by $\gamma E_0^2 a^2 = 4$, which is about 7% larger than minimum critical power.[2,7] If we hold E_0 fixed and minimize z_f with respect to a^2, we find $a^2_{\min} = 8/\gamma E_0^2$; that is to say z_f is a minimum, under conditions of fixed E_0, when the beam radius is adjusted to contain precisely two critical powers.

As a second example, we consider a plane wave. Such a wave has been shown theoretically to be unstable under perturbations of the type[5,8] $\delta A = A_0 u(z)\psi_k(\mathbf{r}_\perp)$, $\delta\phi = -u(z)\psi_k(\mathbf{r}_\perp)$, where A and ϕ are the amplitude and phase, $E = A\exp(i\phi)$, A_0 is the (constant) amplitude of the plane wave, and ψ_k is any solution to the eigenvalue equation $\nabla_\perp^2\psi_k = -k^2\psi_k$. Thus k denotes the wave number

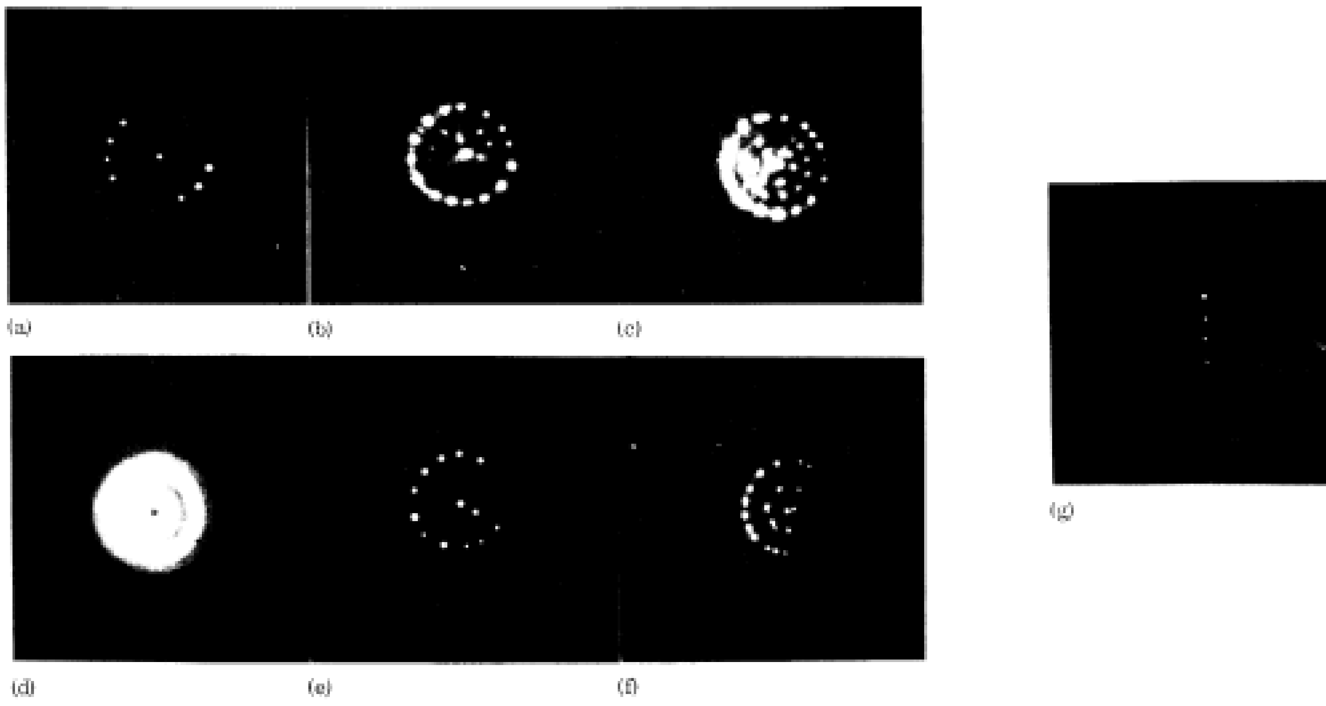


FIG. 2. Experimental photographs showing spatial periodicity of focal spots: (a), (b), and (c) show focal spot patterns for $F=7$ for progressively higher beam powers; (d) is the 6943-Å diffraction pattern of the end of the cell for $F=4$, while (e) and (f) are the corresponding focal spot patterns [(f) forced into aperiodicity by asymmetric intensity distribution)]; (g) is the focal spot pattern for a straight edge diffraction pattern.