# Vector Analysis

## Vector Products

The length (or magnitude) of a vector

$$\mathbf{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \tag{VA-1}$$

is

$$a = |\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}. \tag{VA-2}$$

### Scalar Product (Inner Product, Dot Product)

The dot product of two vectors is

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z = ab\cos(\theta_{\mathbf{ab}}), \tag{VA-3}$$

where $\theta_{\mathbf{ab}}$ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

The length (or magnitude) of $\mathbf{a}$ is

$$a = |\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}. \tag{VA-4}$$

### Cross Product (Outer Product, Vector Product)

The magnitude of the cross product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$|\mathbf{a} \times \mathbf{b}| = ab\sin(\theta_{\mathbf{ab}}). \tag{VA-5}$$

The direction of $\mathbf{a} \times \mathbf{b}$ is determined by the right-hand rule.

A formula for $\mathbf{a} \times \mathbf{b}$ in Cartesian coordinates is

$$\mathbf{a} \times \mathbf{b} = \underbrace{(a_y b_z - a_z b_y)}_{yzx \text{ term}}\,\hat{\mathbf{x}} + \underbrace{(a_z b_x - a_x b_z)}_{zxy \text{ term}}\,\hat{\mathbf{y}} + \underbrace{(a_x b_y - a_y b_x)}_{xyz \text{ term}}\,\hat{\mathbf{z}}. \tag{VA-6}$$

Note that each term is formed from a different cyclic permutation of $xyz$.

Another formula for $\mathbf{a} \times \mathbf{b}$ in Cartesian coordinates is

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}. \tag{VA-7}$$

The triple scalar product of vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}, \tag{VA-8}$$

which is the volume of the parallelepiped bounded by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. It is obvious from this formula and the properties of determinants that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0 = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}). \tag{VA-9}$$

The triple vector product,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}), \tag{VA-10}$$

is orthogonal to both $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$. The BAC-CAB rule evaluates $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ as a linear combination of $\mathbf{b}$ and $\mathbf{c}$:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}). \tag{VA-11}$$

## Equations of Lines and Planes

A point

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{VA-12}$$

lies on a straight line through the point $\mathbf{r}_0$, parallel to the vector $\mathbf{t}$, if and only if

$$\mathbf{t} \times (\mathbf{r} - \mathbf{r}_0) = \mathbf{0}. \tag{VA-13}$$

The point $\mathbf{r}$ lies in a plane through the point $\mathbf{r}_0$, perpendicular to the vector $\mathbf{n}$, if and only if

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0. \tag{VA-14}$$