Impulse response function for Poisson's eq^n

We know that the eq^n $\nabla^2 \Phi = -\frac{\rho}{\epsilon}$

can be solved using an impulse response function, because the Laplacian operator, ∇^2, is linear.

Let the response (i.e., the potential) at $\underset{\sim}{r}$ due to a unit impulse at $\underset{\sim}{r}'$ be $G(\underset{\sim}{r}, \underset{\sim}{r}')$. Then the potential due to an extended source is

$$\Phi(\underset{\sim}{r}) = \Phi_0(\underset{\sim}{r}) - \int G(\underset{\sim}{r}, \underset{\sim}{r}') \frac{\rho(\underset{\sim}{r}')}{\epsilon} dV'$$

where $\nabla^2 G(\underset{\sim}{r}, \underset{\sim}{r}') = \underbrace{\delta(\underset{\sim}{r} - \underset{\sim}{r}')}_{\text{a unit impulse!}}$

defines the impulse response, and Φ_0 is any solution of the homogeneous eq^n (Laplace's eq^n)

$$\nabla^2 \Phi_0 = 0.$$

What is $G(\underset{\sim}{r}, \underset{\sim}{r}')$? We know already that

$$\boxed{G(\underset{\sim}{r}, \underset{\sim}{r}') = \frac{-1}{4\pi|\underset{\sim}{r} - \underset{\sim}{r}'|}}$$

(also known as the <u>Green function</u> for Poisson's eq^n)

Let's verify this impulse response function:

Let $\underset{\sim}{R} = \underset{\sim}{r} - \underset{\sim}{r}' \Rightarrow \nabla G(\underset{\sim}{R}) = -\frac{1}{4\pi} \nabla \frac{1}{R} = \frac{\hat{R}}{4\pi R^2}$.

$$\int_V \nabla \cdot (\nabla G) dV = \left\{ \begin{array}{l} 4\pi R^2 \cdot \frac{1}{4\pi R^2} = 1 \text{ if } V \text{ includes } \underset{\sim}{r}' \\ 0 \text{ if } V \text{ does } \underline{\text{not}} \text{ include } \underset{\sim}{r}' \end{array} \right\} \text{by Gauss' Law}$$

$$\Rightarrow \nabla^2 G = \delta(\underset{\sim}{R}).$$

BOTTOM LINE: IMPULSE RESPONSE F^N $= -\frac{1}{4\pi} \times$ POTENTIAL OF A UNIT POINT CHARGE (FOR POISSON'S EQN).

Spherical wave radiated by a point source,

a.k.a.

Impulse response function for the scalar inhomogeneous Helmholtz eq^n

$$(\nabla^2+k^2)\psi = s(\underset{\sim}{r})$$

where s is a source function:

The impulse response function should satisfy the eq^n

$$(\nabla^2+k^2)G(\underset{\sim}{r},\underset{\sim}{r}') = \delta(\underset{\sim}{r}-\underset{\sim}{r}'),$$

because then the solution of the eq^n $(\nabla^2+k^2)\psi = s$ is

$$\boxed{\psi(\underset{\sim}{r}) = \psi_0(\underset{\sim}{r}) + \int G(\underset{\sim}{r},\underset{\sim}{r}')\, s(\underset{\sim}{r}')\, dV'}$$

where ψ_0 is a solution of the homogeneous eq^n $(\nabla^2+k^2)\psi_0 = 0$.

We'll verify that the impulse response f^n is

$$\boxed{G(\underset{\sim}{r},\underset{\sim}{r}') = -\frac{e^{-jk|\underset{\sim}{r}-\underset{\sim}{r}'|}}{4\pi|\underset{\sim}{r}-\underset{\sim}{r}'|}}$$

To calculate $\nabla^2 \dfrac{e^{-jkR}}{R}$, where $\underset{\sim}{R} = \underset{\sim}{r}-\underset{\sim}{r}'$:

Recall that $\nabla R = \hat{R}$, $\nabla\left(\frac{1}{R}\right) = -\frac{\hat{R}}{R^2}$, $\nabla\cdot\underset{\sim}{R} = 3$

$$\Rightarrow \nabla(e^{-jkR}) = -jk\, e^{-jkR}\, \hat{R}$$

Then

$$\nabla^2 \frac{e^{-jkR}}{R} = \nabla\cdot\left(\nabla \frac{e^{-jkR}}{R}\right)$$

$$= \frac{1}{R}\nabla^2 e^{-jkR} + 2\left(\nabla\frac{1}{R}\right)\cdot\left(\nabla e^{-jkR}\right) + e^{-jkR}\,\nabla^2\left(\frac{1}{R}\right)$$

$$
\begin{aligned}
\nabla^2 e^{-jkR} &= \nabla\cdot\left(\nabla e^{-jkR}\right) \\
&= \nabla\cdot\left(-jk\, e^{-jkR}\, \hat{R}\right) \\
&= \nabla\cdot\left(-jk\,\frac{e^{-jkR}}{R}\, \underset{\sim}{R}\right) \\
&= -jk\left[\underset{\sim}{r}\cdot\nabla\left(\frac{e^{-jkR}}{R}\right) + \frac{e^{-jkR}}{R}\underbrace{\left(\nabla\cdot\underset{\sim}{R}\right)}_{3}\right] \\
&= -jk\left[\left(-jk\,\frac{e^{-jkR}}{R}\,\hat{R} - e^{-jkR}\,\frac{\hat{R}}{R^2}\right)\cdot\underset{\sim}{R} \right. \\
&\qquad\qquad \left. + 3\,\frac{e^{-jkR}}{R}\right] \\
&= -jk\left[-jk\, e^{-jkR} - \frac{e^{-jkR}}{R} + 3\,\frac{e^{-jkR}}{R}\right] \\
&= \left(-k^2 - \frac{2jk}{R}\right) e^{-jkR}
\end{aligned}
$$

Then

$$
\begin{aligned}
\nabla^2\left(\frac{e^{-jkR}}{R}\right) &= \frac{1}{R}\left(-k^2 - \frac{2jk}{R}\right) e^{-jkR} \\
&\quad + 2\left(-\frac{1}{R^2}\,\hat{R}\right)\cdot\left(-jk\, e^{-jkR}\,\hat{R}\right) \\
&\quad - 4\pi\delta(\underset{\sim}{R})\, e^{-jkR} \\
&= -k^2\,\frac{e^{-jkR}}{R} - 4\pi\delta(\underset{\sim}{R})
\end{aligned}
$$

$$
\Rightarrow \boxed{\left(\nabla^2 + k^2\right)\left(-\frac{e^{-jkR}}{4\pi R}\right) = \delta(\underset{\sim}{R}).}
$$

Interpretation & visualization of the spherical wave $\frac{e^{-jkR}}{R}$

We have (from Euler's theorem)

$$\frac{e^{-jkR}}{R} = \frac{\cos kR}{R} - j\,\frac{\sin kR}{R}$$

These real & imaginary parts are standing waves. The real part, $\frac{\cos kR}{R}$, has zeros at $kR = (2n+1)\frac{\pi}{2}$

and maxima near $kR = 2n\pi$

Multiplying the phasor $\frac{e^{-jkR}}{R}$ by $e^{j\omega t}$, one gets

$$\boxed{\frac{e^{j(\omega t - kR)}}{R} = \frac{\cos(\omega t - kR)}{R} + j\,\frac{\sin(\omega t - kR)}{R}}$$

Physical example: Acoustic wave produced by a point source

The real & imaginary part are outwardly propagating spherical waves. The real part, $\frac{\cos(\omega t - kR)}{R}$, has zeros at $\omega t - kR = (2n+1)\frac{\pi}{2}$.

In other words, the R at which a particular zero occurs is

$$kR_n = \omega t - (2n+1)\frac{\pi}{2} \Rightarrow R_n = \frac{\omega}{k}t - (2n+1)\frac{\pi}{2k}.$$

Evidently R_n moves outward with velocity

$$\boxed{v = \frac{\omega}{k}.}$$

It's easy to show that $\frac{e^{j(\omega t + kR)}}{R}$ is an inwardly propagating spherical wave.

Significance of $\frac{1}{R}$ in $\frac{e^{-jkR}}{R}$ and $\frac{e^{j(\omega t-kR)}}{R}$:

The fact that the spherical wave produced by a <u>point</u> source is proportional to $1/R$ is extremely important for understanding radiation.

Suppose that the energy density of the wave $\psi_+ = \psi_0 \frac{e^{j(\omega t-kR)}}{R}$ is $|\psi|^2$ Joules/m^3.

The velocity vector of a surface of constant phase (a wavefront) is

$$\underset{\sim}{V} = v\hat{R} \quad \text{where} \quad v = \frac{\omega}{k}.$$

Then the power flux (energy per m^2 per s) at $\underset{\sim}{R}$ is

$$\underset{\sim}{S} = \underbrace{|\psi|^2}_{J/m^3} \underbrace{v}_{m/s} \hat{R} \qquad J/m^2\text{-}s$$

The energy per unit time passing through a fixed spherical surface of radius a is

$$\int \underset{\sim}{S} \cdot d\underset{\sim}{S} = 4\pi a^2 v \, |\psi|^2 \Big|_{R=a} \qquad J/s$$

$$= 4\pi \frac{\omega}{k} |\psi_0|^2, \text{ \underline{independent of a}}.$$

Comparison with $1/R^2$, etc.

Suppose we had a wave $\phi(R) = \phi_0 \frac{e^{j(\omega t-kR)}}{R^2}$, with energy density $|\phi|^2$ J/m^3. Then the energy/unit time passing through a fixed spherical surface would be

(next page)

$$\left.\begin{array}{l}\text{energy/unit time}\\ \text{through sphere of}\\ \text{radius } a\end{array}\right\} = 4\pi \frac{\omega}{k} |\phi_0|^2 \frac{1}{a^2}$$

since (area of sphere) $\cdot$ (power flux)

$$= 4\pi a^2 \cdot v \, |\phi_0|^2 \frac{1}{a^4} = 4\pi \frac{\omega}{k} |\phi_0|^2 \frac{1}{a^2}.$$

<u>Therefore:</u>

Only waves that vary as $1/R$ at large distances represent radiation.

$\boxed{\begin{array}{l}\underline{\text{Question}}\text{: How can one make a plane wave?} \\ \underline{\text{Partial answer}}\text{: At large distances from the} \\ \quad\text{source, small patches of a spherical wave} \\ \quad\text{are approximately plane.}\end{array}}$

This statement is intuitively clear. Let's see how it follows from the mathematical expression for the spherical wave produced by a point source:

$$\frac{e^{-jkR}}{R} = \frac{e^{-jk|\underset{\sim}{R}_0+\Delta\underset{\sim}{r}|}}{|\underset{\sim}{R}_0+\Delta\underset{\sim}{r}|}$$

$$(|\Delta\underset{\sim}{r}| \ll R_0)$$

$$\approx \frac{e^{-jk|\underset{\sim}{R}_0+\Delta\underset{\sim}{r}|}}{R_0}$$

where

$$|\underset{\sim}{R}_0+\Delta\underset{\sim}{r}| = \left[(\underset{\sim}{R}_0+\Delta\underset{\sim}{r})\cdot(\underset{\sim}{R}_0+\Delta\underset{\sim}{r})\right]^{1/2}$$

$$= \left[R_0^2 + (\Delta\underset{\sim}{r})^2 + 2\,\underset{\sim}{R}_0\cdot\Delta\underset{\sim}{r}\right]^{1/2}$$

$$= R_0\left[1 + \left(\frac{\Delta r}{R_0}\right)^2 + \frac{2}{R_0^2}\,\underset{\sim}{R}_0\cdot\Delta\underset{\sim}{r}\right]^{1/2}$$

$$\approx R_0 + \frac{R_0}{2}\left(\frac{2}{R_0^2}\right)\underset{\sim}{R}_0\cdot\Delta\underset{\sim}{r} + \mathcal{O}\left(\left(\frac{\Delta r}{R_0}\right)^2\right) \quad \text{(binomial expansion)}$$

$$\approx R_0 + \frac{1}{R_0}\,\underset{\sim}{R}_0\cdot\Delta\underset{\sim}{r} = R_0 + \hat{R}_0\cdot\Delta\underset{\sim}{r}$$

$$\Rightarrow \boxed{\frac{e^{-jkR}}{R} \approx \underbrace{\frac{e^{-jkR_0}}{R_0}}_{\text{fixed phasor}} e^{-jk\hat{R}_0\cdot\Delta\underset{\sim}{r}} = \frac{e^{-jkR_0}}{R_0}\,\overbrace{e^{-j\underset{\sim}{k}\cdot\Delta\underset{\sim}{r}}}^{\text{plane wave}}}$$

where $\underset{\sim}{k} = k\hat{R}_0$.